Games in Strategic Form (June 12)

I. Definition of a Game

- "Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare."[1]

A **game in strategic form** is given by $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where

- $N = \{1, 2, \cdots, n\}$: the **set of players**
- S_i: the **set of strategies** of player $i \in N$
- $u_i : \prod_{i \in N} S_i \to \mathbb{R}$: **payoff function** of player $i \in N$

- Notation and Terminology:
  - $\mathbb{R}$: set of real numbers
  - $S := \prod_{i \in N} S_i = S_1 \times S_2 \times \cdots S_n = \{(s_1, s_2, \cdots, s_n) | s_1 \in S_1, \cdots, s_n \in S_n\}$
  - An element $s := (s_1, s_2, \cdots, s_n) \in S$ is sometimes called a **strategy profile** or simply an **outcome** of the game G.

- Assumptions: Each player $i \in N$ chooses some strategy $s_i \in S_i$
  - independently – no communication among players $\to$ noncooperative
  - simultaneously – no advanced knowledge of the strategies chosen by other players

II. Examples of Games in Strategic Form

- Rock, Paper, and Scissors
  - $N = \{1, 2\}$ (2 players, called player 1 and player 2)
  - $S_1 = S_2 = \{R, Pa, Sc\}$ where R denotes "Rock," Pa denotes "Paper," and Sc denotes "Scissors"

[1] These are the opening sentences of Myerson (1991).

– Payoff functions for the 2 players are defined as follows

$$u_1(R,R)=0, u_1(R,Pa)=-1, u_1(R,Sc)=1$$

$$u_1(Pa,R)=1, u_1(Pa,Pa)=0, u_1(Pa,Sc)=-1$$

$$u_1(Sc,R)=-1, u_1(Sc,P)=1, u_1(Sc,Sc)=0$$

and

$$u_2(R,R)=0, u_2(R,Pa)=1, u_2(R,Sc)=-1$$

$$u_2(Pa,R)=-1, u_2(Pa,Pa)=0, u_2(Pa,Sc)=1$$

$$u_2(Sc,R)=1, u_2(Sc,Pa)=-1, u_2(Sc,Sc)=0$$

- Cournot Duopoly
  - $N=\{1,2\}$ (2 players who are called in this setting as firm 1 and firm 2)
  - $S_1=S_2=[0,\infty)$: production level of each firm (strategy sets can be infinite and unbounded)

$$u_1(s_1,s_2)=p(s_1,s_2)s_1-c_1s_1$$

$$u_2(s_1,s_2)=p(s_1,s_2)s_2-c_2s_2$$

where

  - $p(s_1,s_2)=\max\{0,a-(s_1+s_2)\}$ denotes the inverse demand function giving the price of the output when firm 1 produces the amount s_1 and firm 2 produces the amount s_2.
  - c_i: cost per unit production for firm i, assumed to be a positive constant.

III. Formulation using a Matrix and Further Examples

- Rock, paper, and scissors

	R	Pa	Sc
R	$0,0$	$-1,1$	$1,-1$
Pa	$1,-1$	$0,0$	$-1,1$
Sc	$-1,1$	$1,-1$	$0,0$

- Convention
  - Player 1 chooses rows, Player 2 chooses columns.

- The entries represent payoffs in the following form: (player 1's payoff, player 2's payoffs).
- From this point forward, player 1 and player 2 will not be color-coded as in the example above.
- A similar convention applies when the players are called player A and player B so that player A chooses rows and player B chooses columns, and etc.

- Prisoner's dilemma
  - Two people A and B whom the police thinks have committed a crime.
    - If neither A nor B confesses: A and B spend 2 years in jail
    - If A confesses, B does not: A is set free, B spends 6 years in jail
    - If A does not confess, B confesses: A spends 6 years in jail, B is set free
    - If A and B confess: A and B spend 5 years in jail
  - $N = \{A, B\}$
  - Typical Notation: "Not confess" $\rightarrow C$ (for "Cooperate") and "Confess" $\rightarrow D$ (for "Defect"). $S_A = S_B = \{C, D\}$.
  - Define payoffs $= -$(time spent in jail). For example, $u_A(C, C) = -2$.
  - The game can be expressed in the following form, were player A chooses rows and player B chooses columns:

$A \setminus B$	C	D
C	$-2, -2$	$-6, 0$
D	$0, -6$	$-5, -5$

- Chicken game
  - Players A and B drive in separate cars, driving towards each other.
  - Each player chooses whether to turn (C) or to not turn (D)
    - If both A and B turn: A and B do not crash, payoff of zero
    - If A turns and B does not turn: no crash, A is embarassed and B feels brave
    - If A does not turn, and B turns: no crash, A brave, and B embarassed
    - If A and B do not turn: crash
  - Let the payoff associated to being embarassed be -2, feeling brave is 2, and crashing is -5.

$A \setminus B$	C	D
C	$0, 0$	$-2, 2$
D	$2, -2$	$-5, -5$

IV. Mixed Extension of a Game in Strategic Form

Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form where S_i is a finite set for each $i \in N$.

- Each element in S_i is called a **pure strategy** of player i.
- A **mixed strategy** of player i is a function $\sigma_i : S_i \to \mathbb{R}$ such that
  - $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$
  - $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

  where $\sigma_i(s_i)$ indicates the probability that player i plays the strategy s_i.
- $\Delta(S_i)$: the set of mixed strategies of player $i \in N$. (to be explained in further detail)
- Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \prod_{i \in N} \Delta(S_i)$. Under the assumption that players choose their mixed strategies independently, the probability that player 1 plays strategy s_1, player 2 plays s_2, $\cdots$, player n plays s_n is given by

$$\sigma_1(s_1)\sigma_2(s_2) \cdots \sigma_n(s_n) = \prod_{i \in N} \sigma_i(s_i)$$

- The **expected payoff** when each player i chooses a mixed strategy σ_i is given by

$$\pi_i(\sigma_1, \sigma_2, \cdots, \sigma_n) = \sum_{(s_1, s_2, \cdots, s_n) \in S} \left(\prod_{i \in N} \sigma_i(s_i) \right) u_i(s_1, s_2, \cdots, s_n)$$

  where $S := \prod_{i \in N} S_i$.
- $(N, (\Delta(S_i))_{i \in N}, (\pi_i)_{i \in N})$ defines a strategic form game and is called the **mixed extension** of the game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$.

References

Myerson, R. B. (1991). *Game Theory: Analysis of Conflict.* Cambridge: Harvard University Press.