5 Extension 1: Introducing The Public Policies

We now introduce the government into the model. Topics to be covered are

1. Effects of government spending
2. Effects of debt financing
3. Effects of taxations

5.1 Effects of Government Spending under Balanced Budget

Suppose that the government consumes $G(t)$ units of the final good. In per capita terms, $g(t) = G(t)/L(t)$. The government levies lump-sum taxes $T(t)$ to finance the expenditure. Therefore the government's budget constraint is

$$T(t)/L(t) = g(t). \tag{32}$$

We assume the path of $g(t)$ is exogenously given. Then the above equation determines the path of $T(t)$.

Equilibrium The household's flow budget constraint now becomes

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) - T(t)/L(t), \tag{33}$$

The household takes the path of $T(t)$ as given. Therefore the Euler equation does not change and it is found that the dynamics of $c(t)$ is essentially same as that in the economy without the government:

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\dot{c}(t)/c(t) = f'(k(t)) - \delta - \rho.$$

On the other hand, the dynamics of $k(t)$ now becomes

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) - g(t).$$

Figure 3 is the phase diagram when $g(t)$ is exogenously constant over time. Hereafter we assume that $g(t)$ is exogenously constant over time: $g(t) = g$. What happens if g increases? In steady state government spending completely crowds out private consumption, *but has no effect on the capital stock.*

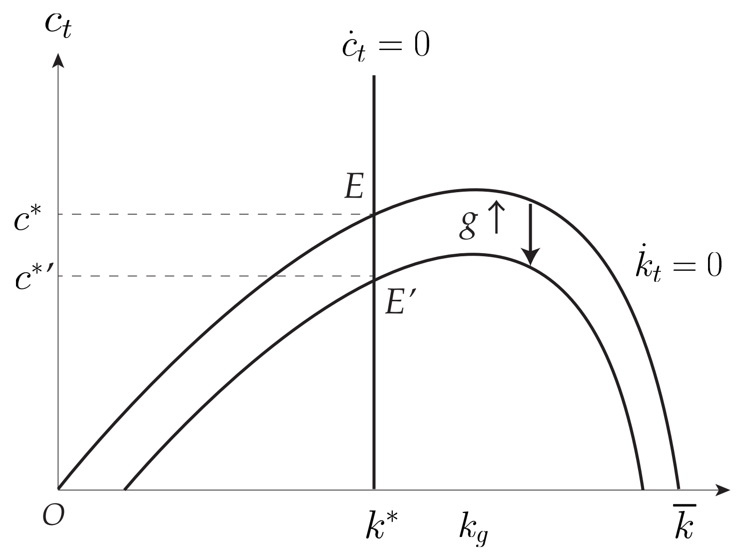

Figure 3: Effects of government spending

5.2 Effects of Debt Financing and the Ricardian Neutrality

Now relax the balanced-budget assumption (32). The government is now allowed to borrow, instead of financing itself only through lump-sum taxes. Let $B(t) \geq 0$ denote stock of government debt at date t. The government's budget constraint is now given by

$$\underbrace{T(t) + \dot{B}(t)}_{\text{Revenue}} = \underbrace{r(t)B(t) + G(t)}_{\text{Expenditure}},$$

or equivalently,

$$\dot{B}(t) = \underbrace{r(t)B(t) + \underbrace{G(t) - T(t)}_{\text{Primary deficit}}}_{\text{Budget deficit}}.$$

Integrating the above equation from zero to infinity,

$$B(0) = \int_0^\infty (T(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt + \lim_{t\to\infty} B(t) \left(-\int_0^t r_s ds\right).$$

The no-Poinzi Game condition which prohibits the government to default is

$$\lim_{t\to\infty} B(t) \left(-\int_0^t r_s ds\right) = 0,$$

which leads the following intertemporal budget constraint of the government:

$$B(0) = \int_0^\infty (T(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{34}$$

The asset market equilibrium is now given by

$$A(t) = K(t) + B(t).$$

Then, the households' aggregate intertemporal budget constraint is

$$\int_0^\infty C(t) \exp\left(-\int_0^t r_s ds\right) dt = k(0) + B(0) + \int_0^\infty (w(t)L(t) - T(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{35}$$

Then, substituting (34) into (35) yields

$$\int_0^\infty C(t) \exp\left(-\int_0^t r_s ds\right) dt = k(0) + \int_0^\infty (w(t)L(t) - G(t)) \exp\left(-\int_0^t r_s ds\right) dt. \tag{36}$$

Notice that

1. Neither taxes $T(t)$ nor the debt $B(t)$ appears in the budget constraint,
2. Only government spending $G(t)$ matters.

This result is summarized as follows:

Proposition 5. *For a given path of $G(t)$, financing it through distortionless taxation and budget deficit are indifferent.*

In other words, the method of finance, whether distortionless taxation or budget deficit has no effect on equilibrium allocation. This property is called the *Ricardian neutrality* (リカードの中立性) or *Ricardian equivalence*(リカードの等価性).

5.3 Effects of Taxations

5.3.1 Effects of Consumption Tax

We now consider the following household' budget constraint:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - (1 + \tau^c(t))c(t) - T(t)/L(t), \tag{37}$$

where $\tau^c(t) \geq 0$ is the rate of consumption tax. The current–value Hamiltonian is

$$H(a(t), c(t), \lambda(t)) = u(c(t)) + \lambda(t)[(r(t) - n)a(t) + w(t) - (1 + \tau^c(t))c(t) - T(t)/L(t)]. \tag{38}$$

The conditions for utility maximization are

$$\partial H/\partial c(t) = 0 \Leftrightarrow u'(c(t)) = \lambda(t)(1 + \tau^c(t)), \tag{39}$$

$$\partial H/\partial a(t) = \dot{\lambda}(t) - (\rho - n)\lambda(t) \Leftrightarrow \dot{\lambda}(t)/\lambda(t) = \rho - r(t), \tag{40}$$

$$\lim_{t\to\infty} \lambda(t)a(t)e^{-(\rho-n)t} = 0. \tag{41}$$

Here assume that $\tau^c(t)$ is constant over time:

$$\dot{\tau}^c(t) = 0.$$

Then, from (39) and (40),

$$\begin{aligned} -\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} &= -\dot{\lambda}(t)/\lambda(t) \\ &= r(t) - \rho. \end{aligned}$$

Thus, the Euler equation does not change.

We can easily obtain the same result even though we extend the *many goods-economy.* Let c_{jt} denote consumption of good $j = 1, 2, \dots, J$. The representative household's utility problem is

$$\begin{aligned} \max \quad & \int_0^\infty e^{-(\rho-n)t} u(c_{1t}, c_{2t}, \dots, c_{Jt}) dt \\ \text{s.t.} \quad & \dot{a}(t) = (r(t) - n)a(t) + w(t) - (1 + \tau^c(t)) \sum_{j-1}^{J} p_{jt}c_{jt} - T(t)/L(t), \\ & a_0 \text{ and NPG}, \end{aligned}$$

where p_{jt} is the price of good j. Without any loss of generality, p_{1t} is normalized to 1.

($*$) Due to the Walras' law, we must normalize the price of one good to unity.

Now (39) is now rewritten as

$$\partial u(\cdot)/\partial c_{jt} = p_{jt}\lambda(t)(1 + \tau^c(t)),\ j = 1, 2, \dots, J.$$

This leads the following well-known formula, which implies the marginal rate of substitution is equal to the relative price:

$$\frac{\partial u(\cdot)/\partial c_{jt}}{\partial u(\cdot)/\partial c_{1t}} = p_{jt},\ j = 2, 3, \dots, J. \tag{42}$$

Thus, the tax rate disappears.

Proposition 6. *An increase in the consumption tax rate has no effect on the equilibrium allocation if*

(i) such an increase applies for all goods, and

(ii) the rate of taxation remains to be flat ($d\tau(t)$ is constant over time).

Caution (Possibility of Static Distortion) If we introduce a "labor-leisure choice" by the household, consumption tax can distort her decision making about her leisure.

→ Exercise.

Caution (Possibility of Dynamic Distortion) If the tax rate on consumption is *anticipated* to increase in the future, the households would want to consume more now and less in the future.

5.3.2 Effects of Capital Income Tax

We now consider the following household' budget constraint

$$\dot{a}(t) = [(1-\tau^a)r(t) - n]a(t) + w(t) - c(t) - T(t)/L(t), \tag{43}$$

where $\tau^a \in [0, 1)$ is the capital income tax rate. It is assumed that the tax rate is constant over time.

The current–value Hamiltonian is now given by

$$H(a(t), c(t), \lambda(t)) = u(c(t)) + \lambda(t)\big\{[(1-\tau^a)r(t) - n]a(t) + w(t) - c(t) - T(t)/L(t)\big\}. \tag{44}$$

The conditions for utility maximization are

$$\partial H/\partial c(t) = 0 \Leftrightarrow u'(c(t)) = \lambda(t), \tag{45}$$

$$\partial H/\partial a(t) = \dot{\lambda}(t) - (\rho - n)\lambda(t) \Leftrightarrow \dot{\lambda}(t)/\lambda(t) = \rho - (1-\tau^a)r(t), \tag{46}$$

$$\lim_{t\to\infty} \lambda(t)a(t)e^{-[(1-\tau^a(t))r(t)-n]t} = 0. \tag{47}$$

Then, from (45) and (46),

$$-\frac{c(t)u''(c(t))}{u'(c(t))}\frac{\dot{c}(t)}{c(t)} = (1-\tau^a(t))r(t) - \rho.$$

To focus on the effects of taxation, we assume that the government does not issue the public bond and $g(t) = 0$. Namely, the government's budget constraint is

$$\tau^a r(t)A(t) + T(t) = 0. \tag{48}$$

This means $T(t) < 0$. The capital income tax revenue is used for the redistribution to the households. Using the same procedure as in section 3, we obtain the dynamic system under capital income taxation:

$$\begin{aligned}
&\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t), \\
&-\frac{c(t)u''(c(t))}{c(t)}\frac{\dot{c}(t)}{c(t)} = (1-\tau^a)f'(k(t)) - \delta - \rho, \\
&\lim_{t\to\infty} k(t)\exp\left(-\int_0^t (f'(k_s) - (n+\delta))ds\right) = 0.
\end{aligned}$$

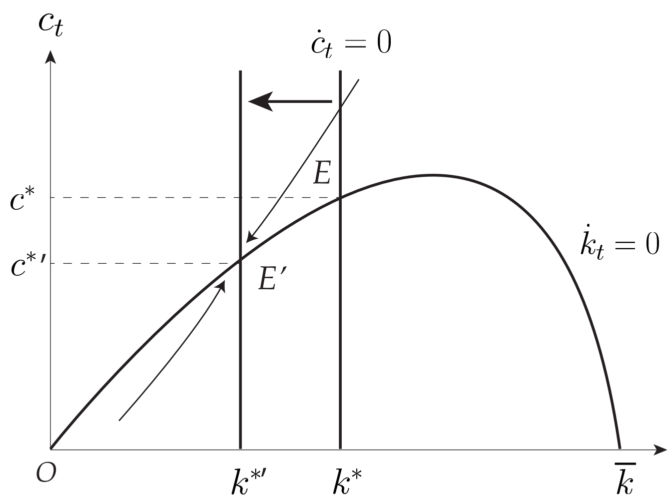

Figure 4: Effets of capital income taxation

Figure 4 shows how the capital income taxation affects the economy. The steady state moves from E to E':

1. the steady state capital stock falls from k^* to $k^{*'}$.

2. the steady state consumption also falls from c^* to $c^{*'}$.

However, note that during the transition, consumption increases temporarily.

6 Extension 2: Introducing Exogenous Technological Progress

In the baseline model, the equilibrium path of $(k(t), c(t))$ converges to the steady state (k^*, c^*). This means that the growth rate of all variables in per capita terms eventually becomes zero. To make the model more realistic, now we introduce the technological progress into the baseline model.

6.1 Labor–augmenting Technological Progress

We extend the production function to

$$Y(t) = F(K(t), Z(t)L(t)),$$

Even if $K(t)$ or $L(t)$ does not change, $Y(t)$ increases if $Z(t)$ increases. Hereafter we interpret $Z(t)$ as the level of technology.

We introduce changes in $Z(t)$ to capture improvements in the technological know-how of the economy.

$$\dot{Z}(t)/Z(t) = \gamma > 0, \tag{49}$$

or equivalently

$$Z(t) = Z(0)\exp(\gamma t). \tag{50}$$

The technological progress such as (49) or (50) is called the *Labor–augmenting Technological Progress* (労働集約的技術進歩).

Define the following new variables:

$$\widetilde{y}(t) \equiv \frac{Y(t)}{Z(t)L(t)}, \quad \widetilde{k}(t) \equiv \frac{K(t)}{Z(t)L(t)}$$

We continue to assume F satisfies Assumptions 3–5, and define the function f as

$$f(\widetilde{k}) \equiv F(\widetilde{k}, 1)$$

The first–order–conditions of profit maximization problem are given by

$$R(t) = f'(\widetilde{k}(t)), \tag{51}$$

$$w(t) = [f(\widetilde{k}(t)) - \widetilde{k}(t)f'(\widetilde{k}(t))]Z(t), \tag{52}$$

which leads

$$R(t)\widetilde{k}(t) + w(t)/Z(t) = \widetilde{y}(t). \tag{53}$$

6.2 Balanced Growth Path

The conditions for utility maximization and the asset market equilibrium does not change from the baseline model. The dynamics of $\widetilde{k}(t)$ is given by

$$\begin{aligned} \dot{\widetilde{k}}(t)/\widetilde{k}(t) &= \dot{k}(t)/k(t) - \gamma \\ \rightarrow \quad \dot{\widetilde{k}}(t) &= f(\widetilde{k}(t)) - (n + \delta + \gamma)k(t) - \widetilde{c}(t), \end{aligned} \tag{54}$$

where $\widetilde{c}(t) \equiv \frac{C(t)}{Z(t)L(t)}$. On the other hand, the dynamics of $\widetilde{c}(t)$ is given by

$$\begin{aligned} \frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} &= \frac{\dot{c}(t)}{c(t)} - \gamma \\ &= \left(-\frac{c(t)u''(c(t))}{u'(c(t))}\right)^{-1} (f'(\widetilde{k}(t)) - \delta - \rho) - \gamma \end{aligned}$$

Thus, $-cu''/u'$ must be constant.

Assumption 7. *$u(c)$ is specified as*

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1 - \theta} & \text{if } \theta > 0, \theta \neq 1, \\ \ln c & \text{if } \theta = 1. \end{cases}$$

Note : $u(c)$ is called the *constant relative risk aversion (CRRA) utility* if it is specified as in the above assumption.

The dynamics of $\widetilde{c}(t)$ is eventually given by

$$\frac{\dot{\widetilde{c}}(t)}{\widetilde{c}(t)} = (1/\theta)(f'(\widetilde{k}(t)) - \delta - \rho - \theta\gamma). \tag{55}$$

Finally, the TVC is reduced to

$$\lim_{t\to\infty} \widetilde{k}(t) \exp\left(-\int_0^t (f'(\widetilde{k}_s) - n - \gamma)ds\right) = 0. \tag{56}$$

(54)–(56) jointly constitute the dynamic system. Since in steady state $\widetilde{k}(t)$ and $\widetilde{c}(t)$ must remain constant, from (54) and (55) we have

$$f'(\widetilde{k}^*) = \rho + \delta + \theta\gamma, \tag{57}$$

and

$$\widetilde{c}^* = f'(\widetilde{k}^*) - (n + \delta + \gamma)\widetilde{k}^*. \tag{58}$$

We can show the unique existence of $(\widetilde{k}^*, \widetilde{c}^*)$ which solves (57) and (58) in a way similar to the model without technological progress. The only additional condition in this case is that because there is growth, we have to make sure that the TVC is in fact satisfied. Substituting (57) into (56), we have

$$\lim_{t\to\infty} \widetilde{k}(t)\exp\{-[\rho - n - (1-\theta)\gamma]t\} = 0.$$

which can only hold if the following assumption is satisfied:

Assumption 8. $\rho - n > (1-\theta)\gamma$.

In the steady state, $\widetilde{k}(t)$ and $\widetilde{c}(t)$ are constant over time. From these definitions,

$$\dot{k}(t)/k(t) = \dot{c}(t)/c(t) = \gamma. \tag{59}$$

Furthermore, since $y(t) = \widetilde{y}(t)Z(t) = f(\widetilde{k}(t))Z(t)$, the growth rate of per capita GDP is γ in the long run.

→ In the steady state, all per capita variables grow at the rate of $\gamma > 0$.

→ In this model, the steady state is called the *Balanced Growth Path* (均斉成長経路).

Proposition 7 (Balanced Growth Path)**.** *In steady state all per capita variable grow at the constant rate of technological progress,* $\gamma > 0$.

7 Summary

- The Ramsey–Cass–Koopmans model is based on the economic agents' intertemporal optimization.
- The competitive equilibrium path in this model corresponds to the social planner's optimal path which achieves the first-best allocation.
- There exists a unique steady state where both of physical capital and consumption are positive.
- Saddle point stability of the steady state means the uniqueness of competitive equilibrium path in this model.

- In the baseline Ramsey model, the government spending crowds out private consumption.
- Consumption tax is distortionless as long as the same rate applies for all goods and it is constant over time.
- Capital income taxation harms the households' savings, thereby capital accumulation. In consequence, the capital stock in the steady state decreases if the tax rate becomes higher.
- By introducing the labor-augmenting technological change to the baseline model, all per capita variables (per capita GDP, capital, consumption...) become to grow at the same constant rate of technological progress in the long-run.

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