

# QIP Course 2: Basics of QIP

Ryutaroh Matsumoto

Nagoya University, Japan

Send your comments to [ryutaroh.matsumoto@nagoya-u.jp](mailto:ryutaroh.matsumoto@nagoya-u.jp)

September 2017

@ Tokyo Tech.

# Acknowledgment and Copyright

This Ph.D course is financially supported by the Villum Foundation through their VELUX Visiting Professor Programme 2013–2014. The slides include figures from the <http://openstaxcollege.org/> and the Wikipedia.

Materials presented here can be reused under the Creative Commons Attribution 4.0 International License

<https://creativecommons.org/licenses/by/4.0>.



# Contents of the slides

- Vector representation of quantum states
- Measurement

# Minimal explanation of quantum mechanics

I will introduce the mathematical model of the quantum theory. It does not include Schrödinger's equation.

Schrödinger's equation is almost always explained in a course on quantum physics. I do not explain that. Schrödinger's equation is required when one wants to know the state of a quantum system as a function of time. We can understand the essential part of QIP without it.

# State of a quantum system

Quantum system: whatever physical phenomenon. E.g. photon polarization.

The state of a quantum system is represented by a complex vector of length (norm) 1 in a complex linear space.

The dimension of the linear space associated with a quantum system is usually infinite dimensional.

Assumption: The dimension of linear space is always finite in this course.

# Notation of vectors

$|\varphi\rangle$ : column vector in the quantum physics  
 $\langle\varphi|$ : the complex conjugate transpose of  $|\varphi\rangle$ .

## Example of states of linear photon polarization

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, | \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$|/\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |\backslash\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The direction of polarization is represented by that of state vector.

## What is represented by a complex vector?

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}$  represents the circular polarization, which means that polarization is rotating as the photon moves.

Measurement of a quantum system = an action of extracting information from the system.

A Hermitian matrix represents how to measure a quantum system.

A complex square matrix  $M$  is *Hermitian* if  $M = M^*$ .

# Eigenvalue and eigenspace

$M$ : complex square matrix

A complex number  $\lambda$  is said to be an *eigenvalue* of  $M$  if there exists a nonzero vector  $\vec{v}$  such that  $M\vec{v} = \lambda\vec{v}$ .

Eigenspace belonging to  $\lambda = \{\vec{u} \mid M\vec{u} = \lambda\vec{u}\}$ .

# Projection onto a subspace

$V$ : linear space

$W$ : subspace of  $V$

$W^\perp$ : orthogonal complement of  $W$  in  $V$

$P_W$ : projection onto  $W$

Any  $\vec{v}$  can be written uniquely as

$$\vec{v} = \vec{w}_1 + \vec{w}_2$$

with  $\vec{w}_1 \in W$  and  $\vec{w}_2 \in W^\perp$ .

$$P_W(\vec{v}) = \vec{w}_1.$$

How to compute the matrix representation of  $P_W$

1 Find an orthonormal basis  $\{|\psi_1\rangle, \dots, |\psi_m\rangle\}$  of  $W$ .

2

$$P_W = |\psi_1\rangle\langle\psi_1| + \dots + |\psi_m\rangle\langle\psi_m| \quad (1)$$

# Spectral decomposition

$M$ : Hermitian matrix

$\lambda_i$ :  $i$ -th eigenvalue of  $M$  ( $\lambda_i \neq \lambda_j$ )

$W_i$ : eigenspace belonging to  $\lambda_i$

$P_i$ : projection onto  $W_i$ .

$$M = \sum_i \lambda_i P_i$$

The above decomposition is called the spectral decomposition of  $M$ .

## How to compute spectral decomposition

- 1 Compute all eigenvalues.
- 2 For each eigenspace  $W_i$ , find an orthonormal basis  $\{|\psi_{i1}\rangle, \dots, |\psi_{im}\rangle\}$  of  $W_i$ .
- 3  $P_i$  is given by

$$P_i = \sum_{k=1}^m |\psi_{ik}\rangle \langle \psi_{ik}|.$$

$\mathcal{H}$ : linear space associated with a quantum system

Measurement is described by an observable  $A$ , which is a Hermitian matrix on  $\mathcal{H}$ .

Results of measuring the observable  $A$  = eigenvalues of  $A$ .

We cannot predict which measurement outcome is obtained before the measurement, e.g. the measurement of polarization. But we can calculate the probability of a measurement outcome.

# Probability of getting a measurement outcome

The quantum system is in state  $|\varphi\rangle$ .

Measuring an observable  $A$ .

$\lambda_1, \dots, \lambda_n$ : eigenvalues of  $A$ .

$$A = \lambda_1 P_1 + \dots + \lambda_n P_n.$$

The probability of getting  $\lambda_i$  as the measurement outcome =

$$\|P_i|\varphi\rangle\|^2. \quad (2)$$

$\alpha$ : complex number with  $|\alpha| = 1$

Since  $|\varphi\rangle$  and  $\alpha|\varphi\rangle$  give the same probability distribution of the measurement outcomes, they are physically indistinguishable.  $|\varphi\rangle$  and  $\alpha|\varphi\rangle$  represent the same quantum state.

# Example of an observable

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since  $Z^* = Z$ , it is Hermitian.

eigenvalue	eigenvector	projector
+1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
-1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The spectral decomposition of  $Z$ :

$$Z = +1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If the state is  $|\varphi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,

$$P_1|\varphi\rangle = |\varphi\rangle, P_2|\varphi\rangle = 0.$$

Probability of getting +1 as the measurement outcome is 1.

Probability of getting -1 as the measurement outcome is 0.

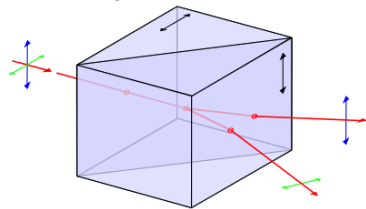
If polarization is represented as in p. 6,  $Z$  represents the measurement by the slit | (or -).

# Nondestructive measurement

When we measure polarization of a photon by a slit, the photon can be absorbed.

There is measurement with which the measured quantum system does not disappear. Such measurement is called *nondestructive measurement*.

Nondestructive measurement of photon polarization can be done by the prism consisting of calcite ( $\text{CaCO}_3$ ) crystal like



Excerpted from

<http://commons.wikimedia.org/wiki/File:Wollaston-prism.svg>.

# State after nondestructive measurement

Quantum state is changed by nondestructive measurement.

Measuring an observable  $A$  of a system with state  $|\varphi\rangle$  nondestructively

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

After getting a measurement outcome  $\lambda_i$ , the state become

$$\frac{P_i |\varphi\rangle}{\|P_i |\varphi\rangle\|}.$$

## Example of nondestructive measurement

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, |\varphi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, |a|^2 + |b|^2 = 1.$$

$$\frac{P_1|\varphi\rangle}{\|P_1|\varphi\rangle\|} = \begin{pmatrix} a/|a| \\ 0 \end{pmatrix}$$

is equivalent to  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , which represents the  $-$  polarization

$$\frac{P_2|\varphi\rangle}{\|P_2|\varphi\rangle\|} = \begin{pmatrix} 0 \\ b/|b| \end{pmatrix}$$

is equivalent to  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , which represents the  $|$  polarization.

Above equations says that after measuring whether the polarization is  $-$  or  $|$ , polarization becomes  $-$  or  $|$  according to the measurement outcome.

## Exercises (60 min.??)

Please discuss them with other students. You are also welcomed to talk with the lecturer.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, |\varphi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

- 1 Is  $X$  a Hermitian matrix?
- 2 Compute the spectral decomposition of  $X$ .
- 3 Suppose that one measures the observable  $X$  of the system with state  $|\varphi\rangle$ . For each measurement outcome, compute its probability and the state after getting the outcome.
- 4 If photon polarization is represented as page 6, which polarizations are measured by  $X$ ?

5 (Optional for non-math students) Prove

$$\sum_{j=1}^n \|P_j|\varphi\rangle\|^2 = 1,$$

where  $P_i$  and  $|\varphi\rangle$  are as defined in Eq. (2). You must not assume that  $\|P_j|\varphi\rangle\|^2$  forms a probability distribution, which you are requested to verify. You must not assume that  $P_j$  can be written as  $|\varphi\rangle\langle\varphi|$  for some vector  $\varphi$ , because an eigenvalue can have two or more linearly independent eigenvectors.

6 (Optional for non-math students). Prove that Eq. (1) is the projection onto  $W$  in the sense of page 9.