Elasticity

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Contents

Introduction

Hook's law for a spring?

$$F = kx$$

Force **spring constant** **displacement**

This class introduces you generalized Hook's law for a crystal (anisotropic materials)

Mechanical response of materials

Elasticity

$$F = kx$$

Visco-Elasticity

Elasto-plasticity

Viscosity

$$F = \eta \dot{x}$$

Plasticity

$$F - F_y = f(x)$$

Visco-Plasticity

3

Deformation of linear elastic isotropic uniform body

Axial loading
: tensile and compression

$$\sigma = E\varepsilon \qquad (1)$$

normal stress **Young's modulus** **normal strain**

Shear loading

$$\tau = G\gamma \qquad (2)$$

shear stress **modulus of rigidity** **shear strain (engineering strain)**

Do you know other two elastic constants?

4

Poisson's ratio v

$$\nu = -\frac{\varepsilon_t}{\varepsilon_\ell}$$

transverse strain

longitudinal strain

Bulk modulus B

$$p = B\frac{\Delta V}{V}$$

volume change

initial volume

isostatic pressure

Among these 4 elastic constants, there are two independent variables. Do you know the reason?

5

Young's modulus of engineering materials

materials	E／GPa
Mild steel	210
Al	70
diamond	~1000
SiC	420
Al_2O_3	390
Si_3N_4	300
ZrO_2	200
glass	70
plolymer	1-3

6

Constitutive equation

Tensor representation

for anisotropic linear elastic body

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (3)$$

stress stiffness strain

This is the generalized Hook's law, containing 9 linear simultaneous equations (3 * 3 for i and j)

We can solve this simultaneous equations for ε_{kl}, and then……

7

Constitutive equation

Tensor representation

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \qquad (4)$$

strain compliance stress

Note that the symbol do not correspond to the name.

We can express these tensor equation in several ways.

8

Constitutive equation

Tensor representation by matrix (not familiar)

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{21} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1132} & C_{1131} & C_{1113} & C_{1112} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2232} & C_{2231} & C_{2213} & C_{2212} & C_{2221} \\ & & & & \vdots & & & & \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2332} & C_{2331} & C_{2313} & C_{2312} & C_{2321} \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ & & & & \vdots & & & & \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2132} & C_{2131} & C_{2113} & C_{2112} & C_{2121} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{pmatrix} \quad (5)$$

9 * 9 = 81 C_{ijkl} components nominally

9

Constitutive equation

But there are the relations (symmetry) between C_{ijkl}, as below

$$C_{ijkl}=C_{jikl}=C_{ijlk}=C_{klij}$$

So, we have 21 independent components in C_{ijkl}

C_{ijkl} is the forth rank tensor. Do you know the reason?

10

Constitutive equation

Assume a new Cartesian coordinate system is set by rotation of the old one around the origin

old σ_{ij} ε_{ij}

new σ'_{mn} ε'_{ts}

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (6)$$

$$\varepsilon_{kl} = v_{kt}v_{ls}\varepsilon'_{ts} \qquad (7)$$

$$\sigma'_{mn} = v_{mi}v_{nj}\sigma_{ij} \qquad (8)$$

$$\sigma'_{mn} = v_{mi}v_{nj}C_{ijkl}v_{tk}v_{sl}\varepsilon'_{ts}$$

$$\therefore C'_{mnts} = v_{mi}v_{nj}v_{tk}v_{sl}C_{ijkl} \qquad (9)$$

v_{kt} **Direction cosines of axis from old to new system**

11

Constitutive equation

Matrix representation (engineering strain form)

Stress and strain tensor are symmetrical tensors, and therefore, they have 6 independent components.

$$\sigma_{ij} = \sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \varepsilon_{kl} = \varepsilon_{kl} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

12

Constitutive equation

Matrix representation (engineering strain form)

Stress and strain tensor are expressed by 6-dimensional vector as below,

$$\sigma_{11} \to \sigma_1 \qquad \varepsilon_{11} \to \varepsilon_1$$
$$\sigma_{22} \to \sigma_2 \qquad \varepsilon_{22} \to \varepsilon_2$$
$$\sigma_{33} \to \sigma_3 \qquad \varepsilon_{33} \to \varepsilon_3$$
$$\sigma_{23} = \sigma_{32} \to \sigma_4 \qquad \varepsilon_{23} = \varepsilon_{32} \to \frac{1}{2}\varepsilon_4 \quad (\varepsilon_4 = 2\varepsilon_{23})$$
$$\sigma_{31} = \sigma_{13} \to \sigma_5 \qquad \varepsilon_{31} = \varepsilon_{13} \to \frac{1}{2}\varepsilon_5 \quad (\varepsilon_5 = 2\varepsilon_{31})$$
$$\sigma_{12} = \sigma_{21} \to \sigma_6 \qquad \varepsilon_{12} = \varepsilon_{21} \to \frac{1}{2}\varepsilon_6 \quad (\varepsilon_6 = 2\varepsilon_{12})$$

13

Constitutive equation

Matrix representation (engineering strain form)

In tensor representation, ε_{23} and ε_{32} are added separately, but matrix representation, ε_4 is added once, so, the value of ε_4 should be double of ε_{23}, and ε_4 is the same as engineering shear strain γ_{23}

$$\varepsilon_{23} = \varepsilon_{32} \to \frac{1}{2}\varepsilon_4 \qquad (\varepsilon_4 = 2\varepsilon_{23} = \gamma_{23})$$

14

Constitutive equation

Matrix representation (engineering strain form)

$$\sigma_m = C_{mn}\varepsilon_n \qquad (10)$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} \qquad (11)$$

6 * 6 = 36 C_{mn} components nominally 15

Constitutive equation

But there are the relations (symmetry) between C_{mn}, also

$$C_{mn} = C_{nm}$$

So, we can undersntand why 21 independent components in C_{ijkl} (21=6+5+4+3+2+1)

This comes from elastic strain energy can be expressed by positive quadratic function of strain

$$W = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

16

Constitutive equation

So, matrix representation is

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & Sym. & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{pmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{pmatrix} \qquad (12)$$

17

Constitutive equation

In the same way, compliance tensor can be

$$\varepsilon_m = S_{mn}\sigma_n \qquad (13)$$

Note that the next transformation is needed to obtain S_{mn}

$$\begin{cases} S_{ijkl} = S_{mn} & (Both\_m\_and\_n\_=1,2,3) \\ S_{ijkl} = \frac{1}{2}S_{mn} & (Either\_m\_or\_n\_=4,5,6) \\ S_{ijkl} = \frac{1}{4}S_{mn} & (Both\_m\_and\_n\_=4,5,6) \end{cases} \qquad (14)$$

18

Constitutive equation

You can understand this when you write it down from tensor to matrix representation,

We can see the same S_{16} term, so originally, S_{16} should be half

$$\begin{aligned}\varepsilon_{11} &= S_{1111}\sigma_{11} + S_{1112}\sigma_{12} + S_{1113}\sigma_{13} \\ &+ S_{1121}\sigma_{21} + S_{1122}\sigma_{22} + S_{1123}\sigma_{23} \\ &+ S_{1131}\sigma_{31} + S_{1132}\sigma_{32} + S_{1133}\sigma_{33}\end{aligned} \rightarrow \begin{aligned}\varepsilon_{1} &= S_{11}\sigma_{1} + \frac{1}{2}S_{16}\sigma_{6} + \frac{1}{2}S_{15}\sigma_{5} \\ &+ \frac{1}{2}S_{16}\sigma_{6} + S_{12}\sigma_{2} + \frac{1}{2}S_{14}\sigma_{4} \\ &+ \frac{1}{2}S_{15}\sigma_{5} + \frac{1}{2}S_{14}\sigma_{4} + S_{13}\sigma_{3}\end{aligned} \qquad (15)$$

19

Constitutive equation

You can understand this when you write it down from tensor to matrix representation,

In the matrix representation, strain ε_{23} is related to half of ε_4 initially.

And we can see the same S_{46} term, so originally, S_{46} should be 1/4

$$\begin{aligned}\varepsilon_{23} &= S_{2311}\sigma_{11} + S_{2312}\sigma_{12} + S_{2313}\sigma_{13} \\ &+ S_{2321}\sigma_{21} + S_{2322}\sigma_{22} + S_{2323}\sigma_{23} \\ &+ S_{2331}\sigma_{31} + S_{2332}\sigma_{32} + S_{2333}\sigma_{33}\end{aligned} \rightarrow \begin{aligned}\frac{1}{2}\varepsilon_{4} &= \frac{1}{2}S_{41}\sigma_{1} + \frac{1}{4}S_{46}\sigma_{6} + \frac{1}{4}S_{45}\sigma_{5} \\ &+ \frac{1}{4}S_{46}\sigma_{6} + \frac{1}{2}S_{42}\sigma_{2} + \frac{1}{4}S_{44}\sigma_{4} \\ &+ \frac{1}{4}S_{45}\sigma_{5} + \frac{1}{4}S_{44}\sigma_{4} + \frac{1}{2}S_{43}\sigma_{3}\end{aligned} \qquad (16)$$

20

Origin of elasticity

Elasticity means that shape and dimension of a body return back to the original state after unloading.

Energy elasticity

metal and ceramics
stiffness decreases when it is heated.

Entropy elasticity

polymer
stiffness increases when it is heated.

21

Origin of elasticity

Thermodynamics 1st law

$$dU = \delta Q + \sigma d\varepsilon \qquad (26)$$

From the 2nd law, in the reversible process, entropy change per unit volume can be expressed by

$$dS = \frac{\delta Q}{T} \qquad (27)$$

By combining these two equations, we obtain for the isothermal reversible process,

22

Origin of elasticity

$$\sigma = \left(\frac{\partial U}{\partial \varepsilon}\right)_T - T\left(\frac{\partial S}{\partial \varepsilon}\right)_T \qquad (28)$$

The first term is the stress by increase of internal energy U with increasing the strain, and the second term is the stress by decrease in entropy S .

23

Origin of elasticity

On the other hand, Helmholz energy F per unit volume is,

$$F = U - TS \qquad (29)$$

The infinitesimal change of F is

$$\begin{aligned} dF &= dU - TdS - SdT \\ &= \delta Q + \sigma d\varepsilon - TdS - SdT \qquad (30) \end{aligned}$$

24

Origin of elasticity

Considering isothermal condition dT=0, Eq.(30) gives,

$$\left(\frac{\partial F}{\partial \varepsilon}\right)_T = \sigma \qquad (31)$$

And if we consider temperature changes in quasi-static by keeping strain constant,

$$\left(\frac{\partial F}{\partial T}\right)_\varepsilon = -S \qquad (32)$$

25

Origin of elasticity

From Eq.(31) and (32) we have,

$$\left(\frac{\partial \sigma}{\partial T}\right)_\varepsilon = \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial \varepsilon}\right) = \frac{\partial^2 F}{\partial T \partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial F}{\partial T}\right) = -\left(\frac{\partial S}{\partial \varepsilon}\right)_T \qquad (33)$$

Substitute this relation into Eq.(28), so,

$$\sigma = \left(\frac{\partial U}{\partial \varepsilon}\right)_T + T\left(\frac{\partial \sigma}{\partial T}\right)_\varepsilon \qquad (34)$$

26

Origin of elasticity

Experimental results showed that 8% vulcanized gum(sulfur added) had a relation σ ＝ cT when it was heated.

Look back Eq.(28), Eq.(28) should be,

$$\sigma = 0 - T\left(\frac{\partial S}{\partial \varepsilon}\right)_T \qquad (35)$$

And corresponding to σ ＝ cT results, the 2nd term $\left(\partial S / \partial \varepsilon\right)_T$ in Eq.(35) should be negative.

27

Origin of elasticity

If is is so, from Eq.(33) $\left(\frac{\partial \sigma}{\partial T}\right)_\varepsilon = \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial \varepsilon}\right) = \frac{\partial^2 F}{\partial T \partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial F}{\partial T}\right) = -\left(\frac{\partial S}{\partial \varepsilon}\right)_T \quad (33)$**, we obtain,**

$$\left(\frac{\partial \sigma}{\partial T}\right)_\varepsilon > 0 \qquad (36)$$

This means that the stress should be increased when temperature raises under the condition of constant strain.

Young's modulus increases when temperature raises. (Entropy elasticity)

28

Origin of elasticity

For energy elasticity, from Eq.(28), we can see,

$$\sigma = \left(\frac{\partial U}{\partial \varepsilon} \right)_T - 0 \qquad (37)$$

The work done by external load is stored as the increase in internal energy by the change in distance and angle between atoms.

Young's modulus decreases when temperature raises.

29

Entropic Elasticity of an Oxide Glass

Seiji INABA (Asahi Glass), Hideo HOSONO (Tokyo Tech.), Setsuro ITO (Asahi Glass)

Recently we revealed that mixed alkali metaphosphate glass with the composition of $Li_{0.25}Na_{0.25}K_{0.25}Cs_{0.25}PO_3$ mol% showed large anisotropy and highly orientated P-O-P chain structure when the glass was deformed under uniaxial stress above glass transition temperature and the deformed structure was frozen by cooling under stress. In the present study we found the anisotropic glass showed a huge thermal shrinkage of 35% in length of glass when heated at deformation point (η=10 9.7 Pa.s). We will discuss the shrinkage mechanism in terms of viscoelastic properties.

30