Weibull statistics

Kouichi Yasuda

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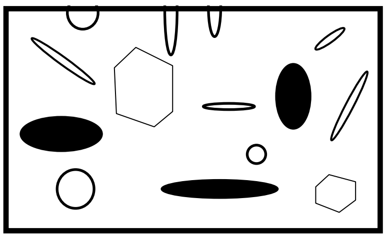
Many types of defects with different size

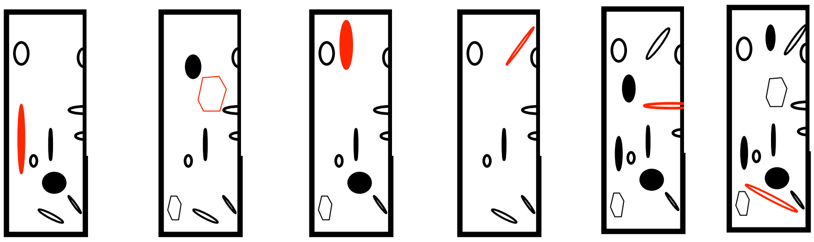
Actual ceramics contain **many types of defect**, such as **crack-like defect**, **pore**, **abnormally grown large grain**, **impurity**, **grinding damage** etc.

And **each defect has its own size distribution,** also. It means....



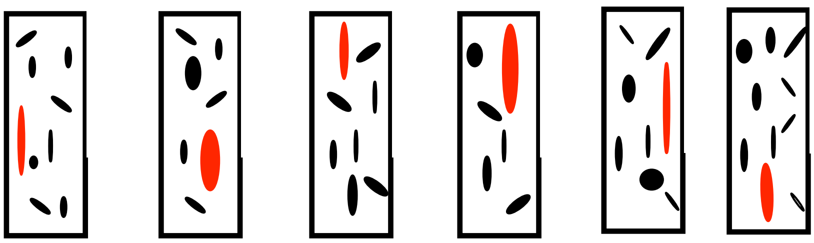
If we make many Al_2O_3 specimens, each specimen has many types of defects with different size.

Therefore, each specimen has its own strength depending on defect size. Wide strength distribution.



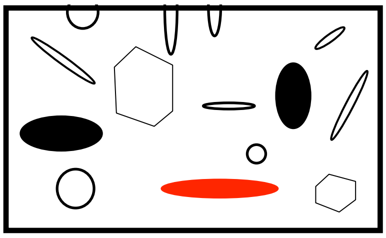
This is the real situation,

But in order to simplify the discussion, we begin with single defect case, we call single mode Weibull distribution first.



Single mode Weibull Distribution

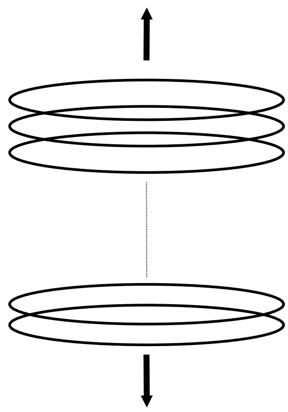
Similar to ceramics, there are a lot of defects inside, and by the weakest defect, fracture begins. We call this concept "Weakest link model"



Weakest link model is explained by series of rings.

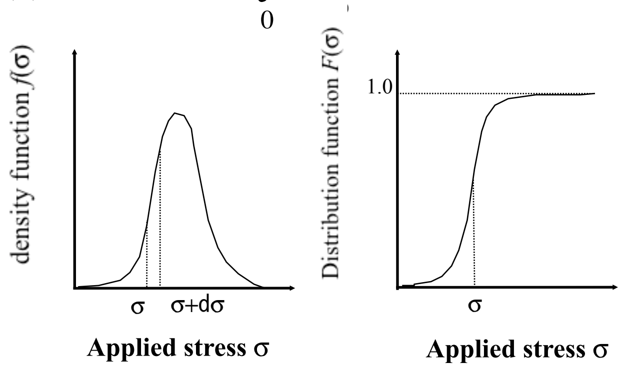
These rings have their own strength distribution.

But this link is fractured when the weakest ring is broken.



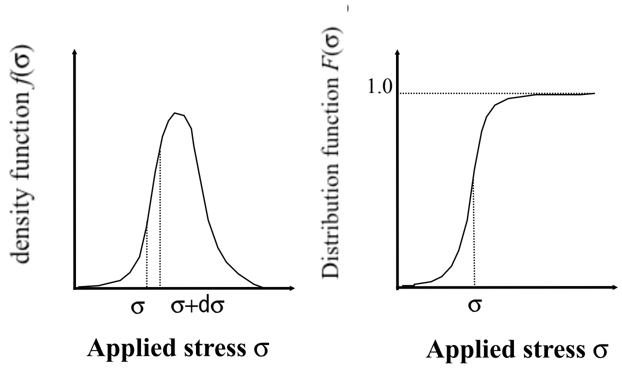
Before the discussion on single mode Weibull distribution, we define distribution function F(σ) and density function f(σ) in statistics.

$$F(\sigma)=\int_0^{\sigma} f(t)dt \qquad (1)$$



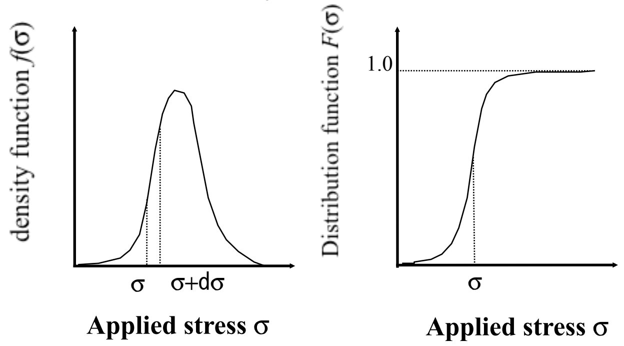
f(σ)dσ is the probability that fracture occurs at the applied stress σ～σ +dσ.

$$F(\sigma)=\int_0^{\sigma} f(t)dt \qquad (1)$$



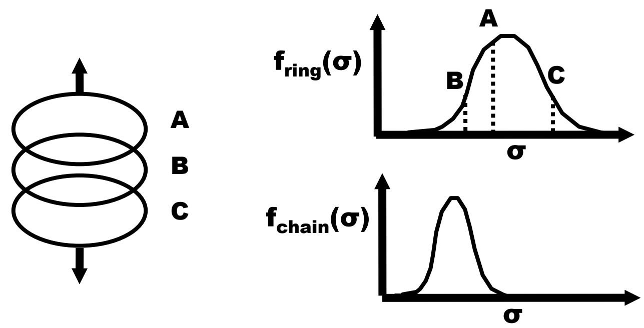
F(σ) is the probability that fracture occurs at the applied stress 0～σ.
From this, F(σ) is called fracture probability

$$F(\sigma) = \int_0^\sigma f(t)dt \qquad (1)$$

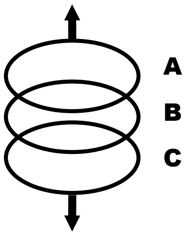


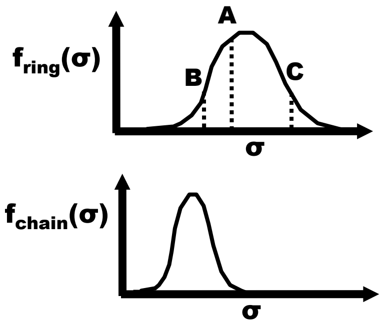
When each ring's strength scatters depending on $f_{ring}(\sigma)$

So, we arbitrary select 3 strength values for A,B,C rings, and then make the chain by the rings A,B,C.



In this case, the strength of the chain is determined by ring B's strength.
We do the same process, we find the strength distribution $f_{chain}(\sigma)$ of the chain around the left end of $f_{ring}(\sigma)$

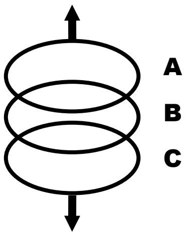


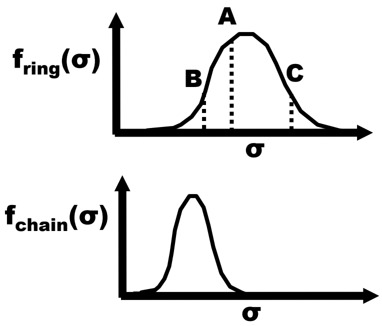


To express this in mathematics,

$$1-F_{chain}(\sigma)=\left\{1-F_{ring}(\sigma)\right\}\left\{1-F_{ring}(\sigma)\right\}\left\{1-F_{ring}(\sigma)\right\} \qquad (2)$$

This means the chain does not break at some stress σ when ring A is not broken, ring B is not broken, and ring C is not broken.





Generally, we obtain

$$1-F_{chain}(\sigma)=\left\{1-F_{ring}(\sigma)\right\}^{n} \qquad (3)$$

In the weakest link model, lower side of $f_{ring}(\sigma)$ determines fracture, so $f_{ring}(\sigma) << 1$. We approximate,

$$1-F_{chain}(\sigma)=\exp\left[\log\left\{1-F_{ring}(\sigma)\right\}^{n}\right]$$

$$\cong \exp\left\{-nF_{ring}(\sigma)\right\}$$

$$\therefore F_{chain}(\sigma)=1-\exp\left\{-nF_{ring}(\sigma)\right\} \qquad (4)$$

$$F_{chain}(\sigma)=1-\exp\left\{-nF_{ring}(\sigma)\right\}$$

From Eq.(4), we find $F_{chain}(\sigma)$ depends on not only $F_{ring}(\sigma)$ but also n.

This expresses volume effect in brittle fracture (larger body has lower strength).

When n becomes infinitely large, $F_{chain}(\sigma)$ is approaching to three types of asymptotic functions.

The third asymptotic function is Weibull distribution.

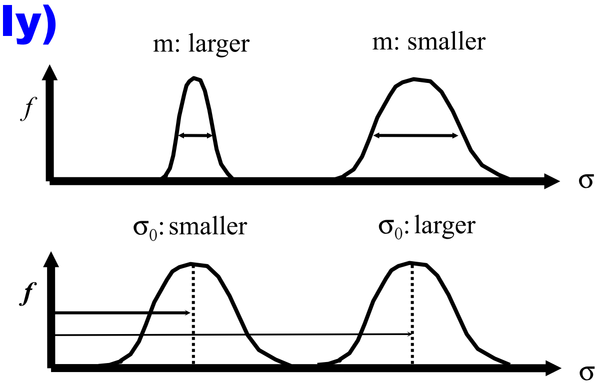
$$F(\sigma) = 1 - \exp\left\{-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right\} \qquad (5)$$

$$f(\sigma) = \frac{V}{V_0}\frac{m\sigma^{m-1}}{(\sigma_0)^m}\exp\left\{-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right\} \qquad (6)$$

m is shape parameter, σ_0 is scale parameter, V is specimen volume, V_0 is unit volume. V/V_0 is the same meaning of n (volume effect included)

The shape parameter m is larger, strength distribution is not scattered.

The scale parameter σ_0 is almost the same as the average strength(not exactly)



The next problem is how to estimate these parameters from strength data.

We have two methods,

(1)Weibull plot (graphical method)

(2)Maximum likelihood method (numerical method)

(1)Weibull plot (graphical method)

$$F(\sigma) = 1 - \exp\left\{ -\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m \right\} \qquad (5)$$

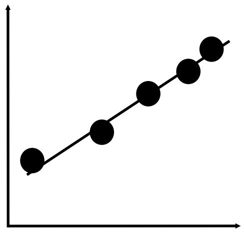
By making 1-F(σ), and then taking logarithm twice, we obtain,

$$\ln\ln\frac{1}{1-F(\sigma)} = m\ln\sigma + \ln\frac{V}{V_0\left(\sigma_0\right)^m} \qquad (7)$$

From this equation, by plotting $\ln\ln\frac{1}{1-F(\sigma)}$ **against** $\ln\sigma$ **, we obtain straight line.**

$$\ln\ln\frac{1}{1-F(\sigma)} = m\ln\sigma + \ln\frac{V}{V_0(\sigma_0)^m} \qquad (7)$$

By regression analysis, we obtain m from the slope and σ_0 from y-intercept.



We do this plot actually.

We have tensile tests on n specimens, and get strength dataset $\{\sigma_f^i\}\ (i=1,n)$

But we can not do Weibull plot with strength data σ_f^i **only. We should assign fracture probability F_i to each strength.**

To do so, we use mean rank method

$$F_i = \frac{i}{n+1} \qquad (8)$$

Here, i is the ordered number when strength data are sorted in ascending order.

If we get $\{\sigma_f^i\}$ **={101MPa, 97MPa, 105MPa},**

σ_f^i	101MPa,	97MPa,	105MPa
i	2,	1,	3
F_i	$\frac{2}{3+1}=0.5,$	$\frac{1}{3+1}=0.25,$	$\frac{3}{3+1}=0.75$

It means the mean rank method gives equally spaced fracture probability to each strength data.

So, now we obtain {F_i**,** σ_f^i**} (i=1,n) datapoints, then get Weibull plot to estimate m and** σ_0**.**

This is Weibull plot method.

(2) maximum likelihood method

As shown, *$f(\sigma^*)d\sigma$* means the probability that fracture occurs when applied stress is between σ^* and $\sigma^*+d\sigma$

In place of σ^*, we use the actual strength σ_f^i *$f(\sigma_f^i)d\sigma$* means the probability that fracture occurs when applied stress is between σ_f^i and $\sigma_f^i+d\sigma$

So, the probability that we get n strength dataset ($\sigma_f^1, \sigma_f^2, \sigma_f^3, \cdots, \sigma_f^n$) is,

$$\prod_{i=1}^{n} f(\sigma_f^i)\underbrace{d\sigma\cdots d\sigma}_{n} = \prod_{i=1}^{n} f(\sigma_f^i : m, \sigma_0)\underbrace{d\sigma\cdots d\sigma}_{n}$$

$$\equiv L(m, \sigma_0 : \sigma_f^i)\underbrace{d\sigma\cdots d\sigma}_{n} \qquad (9)$$

This L function is newly defined as likelihood function.

And originally, L has arguments σ_f^i
But we we regard L as a function of m and σ_0.

If we think so, strength dataset ($\sigma_f^1, \sigma_f^2, \sigma_f^3, \cdots, \sigma_f^n$) should appear at the maximum of L($\hat{m}$, $\hat{\sigma}_0$) .

And these $\hat{m}$ and $\hat{\sigma}_0$ must be mostly close to the true values of m and σ_0.

This is the concept of maximum likelihood method.

To obtain maximum of L, we solve,

$$\left.\begin{aligned} \frac{\partial \ln L}{\partial m} &= 0 \\ \frac{\partial \ln L}{\partial \sigma_0} &= 0 \end{aligned}\right\} \qquad (10)$$

then, actually we get

$$\frac{n}{m} + \sum_{i=1}^{n} \ln \sigma_f^i - \frac{n \sum_{i=1}^{n} \left(\sigma_f^i\right)^m \ln \sigma_f^i}{\sum_{i=1}^{n} \left(\sigma_f^i\right)^m} = 0 \qquad (11)$$

$$\sigma_0 = \left\{ \frac{1}{n} \left(\frac{V}{V_0} \right) \sum_{i=1}^{n} (\sigma_f^i)^m \right\}^{\frac{1}{m}} \qquad (12)$$

This equation should be solved by Newton method(numerical method) because Eq.(11) is a non-linear equation.

This is the basic concept of single mode Weibull distribution.

Competing mode Weibull Distribution

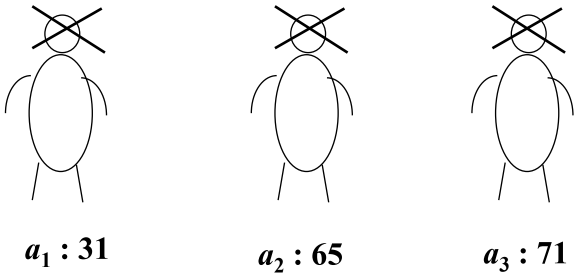
Now we consider the case that there are many fracture causes and each defect competes with other defects.

In this case we use competing mode Weibull dsitribution.

At first, we look at death of human.

There is a fisherman village A. Village people always eat salty food, and they die by brain bleeding only.
For example, a_1 died at 31, a_2 65, a_3 71

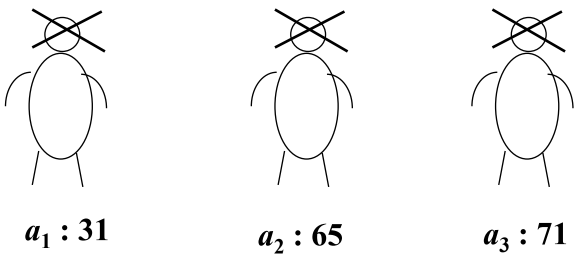
(a) Village A : die by brain bleeding only



In this case, we can express death probability as single mode Weibull distribution. y is the age.

$$F_A(y) = 1 - \exp\left\{ -\left(\frac{y}{y_{0A}} \right)^{m_A} \right\} \qquad (13)$$

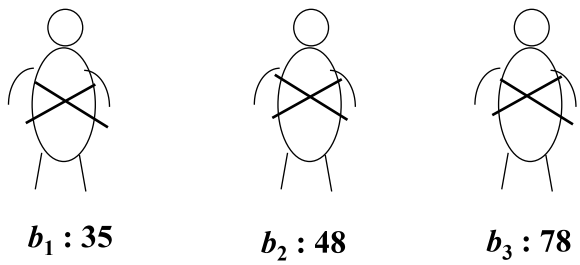
(a) Village A : die by brain bleeding only



There is another hunter village B. Village people always eat meet, and they die by heart attack only.

For example, b_1 died at 35, b_2 48, b_3 78

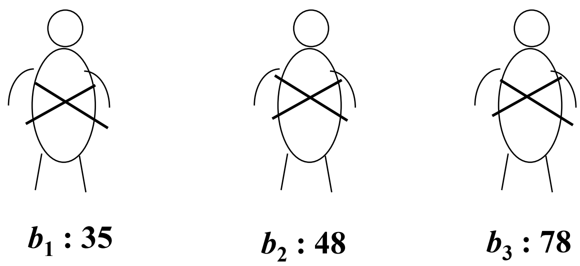
(b) Village B : die by heart attack only



In this case also, we can express death probability as single mode Weibull distribution. y is the age.

$$F_B(y) = 1 - \exp\left\{ -\left(\frac{y}{y_{0B}} \right)^{m_B} \right\} \qquad (14)$$

(b) Village B : die by heart attack only



Later, both villages are united as one big village AB.
At first, both people do not change their own food culture.

People from A always die in brain bleeding only.

People from B always die in heart attack only.

In this case, death probability $F_{AB}(y)$ is,

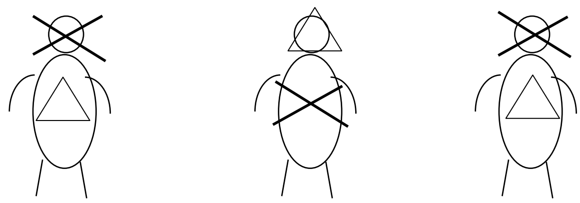
$$F_{AB}(y) = p_A\left[1-\exp\left\{-\left(\frac{y}{y_{0A}}\right)^{m_A}\right\}\right] + p_B\left[1-\exp\left\{-\left(\frac{y}{y_{0B}}\right)^{mB}\right\}\right] \qquad (15)$$

Here, p_A is the population of A, and p_B is that of B.

This is the mixed mode Weibull distribution where each person has only one death cause but those people are mixed.

Finally, AB village becomes city C. Food cultures are mixed, and people C eat both salty food and meet, and have both death causes.
For example, c_1 died at 42(B>H), b_2 48(H>B), b_3 78(B>H)

(C) City C : die by both death causes

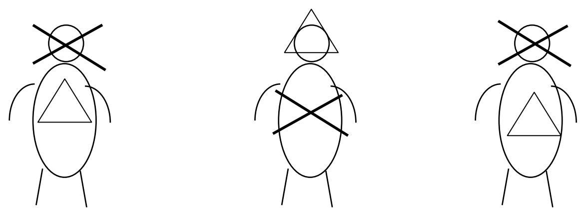


c_1 : 42 c_2 : 55 c_3 : 65

In this case, we can express death probability as competing mode Weibull distribution. y is the age.

$$F_C(y) = 1 - \exp\left\{ -\left(\frac{y}{y_{0A}} \right)^{m_A} - \left(\frac{y}{y_{0B}} \right)^{m_B} \right\} \qquad (16)$$

(C) City C : die by both death causes



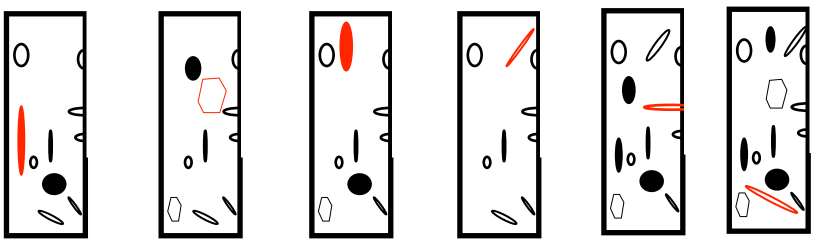
c_1 : 42 c_2 : 55 c_3 : 65

This equation means the survive at some age is the product event that a person does not die by both brain bleeding and heart attack.

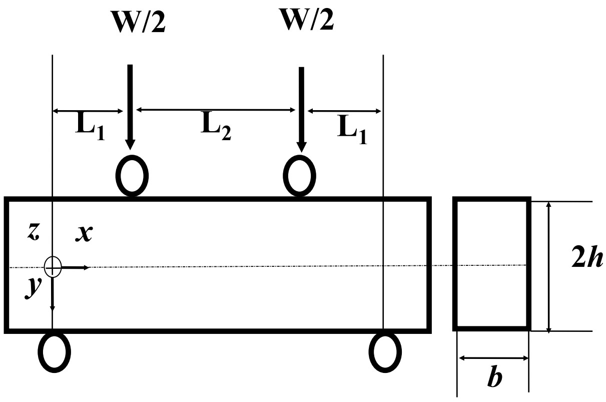
$$1-F_C(y)=\exp\left\{-\left(\frac{y}{y_{0A}}\right)^{m_A}\right\}\exp\left\{-\left(\frac{y}{y_{0B}}\right)^{m_B}\right\}$$

That is, this is totally different from mixed mode Weibull distribution.

As explained before, one ceramic specimen has many fracture causes and many defects. So, to express strength distribution of ceramics, we should use competing mode Weibull distribution.

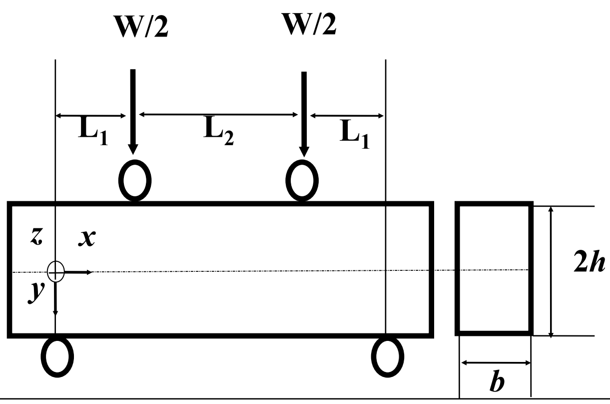


Now we consider 4-point bending test, and apply competing mode Weibull distribution to it.



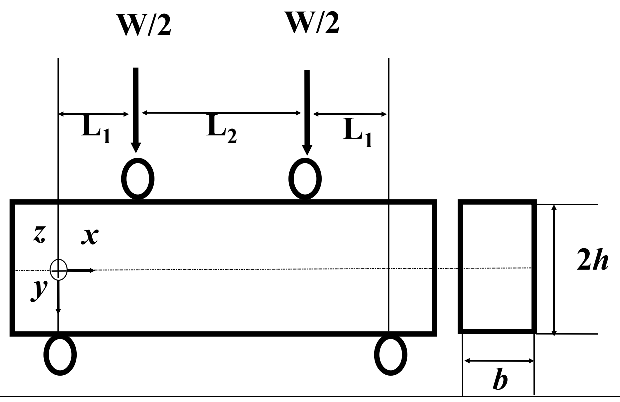
We use (x,y,z) coordinate below, the stress $\sigma_{xx}(x,y,z)$ is given when applied load is W.

$$\sigma_{xx}(x,y,z)=\sigma_m h_1(x) h_2(y) h_3(z) \tag{17}$$



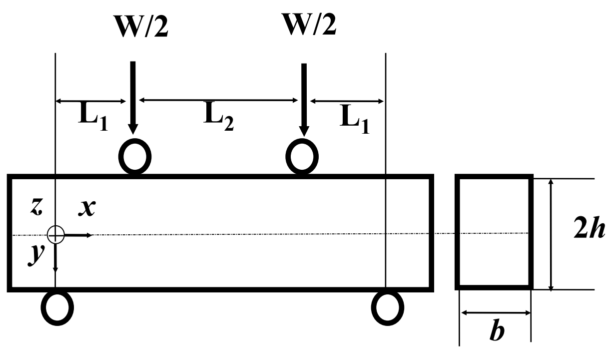
Here, σ_m is the maximum stress in the specimen given by,

$$\sigma_m = \frac{3WL_1}{4bh^2} \qquad (18)$$



Usually, bending strength σ_f is calculated with this equation by substituting fracture load W_f in place of W

$$\sigma_f = \frac{3W_f L_1}{4bh^2} \qquad (18)$$

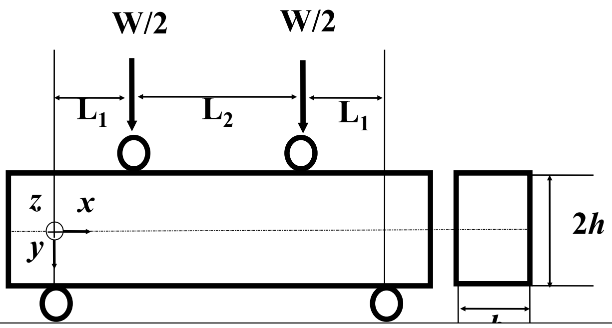


Functions h_i are

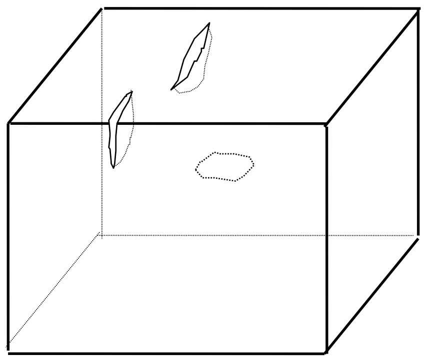
$0 \leqq x < L_1$ $$h_1(x)=\frac{x}{L_1},\quad h_2(y)=\frac{y}{h},\quad h_3(z)=1 \tag{19}$$

$L_1 \leqq x < L_1+L_2$ $$h_1(x)=1,\quad h_2(y)=\frac{y}{h},\quad h_3(z)=1 \tag{20}$$

$L_1+L_2 \leqq x \leqq 2L_1+L_2$ $$h_1(x)=\frac{2L_1+L_2-x}{L_1},\quad h_2(y)=\frac{y}{h},\quad h_3(z)=1 \tag{21}$$



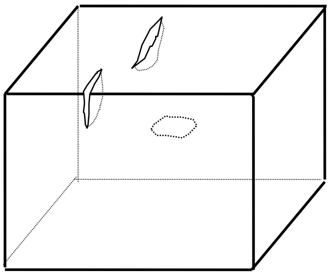
For fracture causes, we assume three causes, inner defect, surface defect, edge defect.



In this case, F(σ) and f(σ) are

$$F(\sigma) = 1 - \exp(-B_1 - B_2 - B_3) \qquad (22)$$

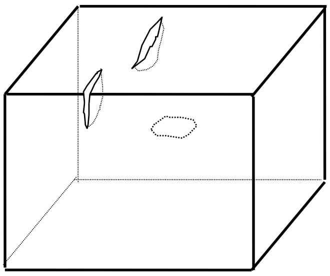
$$f(\sigma) = \left(\frac{\partial B_1}{\partial \sigma} + \frac{\partial B_2}{\partial \sigma} + \frac{\partial B_3}{\partial \sigma}\right)\exp(-B_1 - B_2 - B_3) \qquad (23)$$



Here, B_i is called risk of rapture. i=1(inner), i=2(surface), i=3(edge).

$$F(\sigma) = 1 - \exp(-B_1 - B_2 - B_3) \qquad (22)$$

$$f(\sigma) = \left(\frac{\partial B_1}{\partial \sigma} + \frac{\partial B_2}{\partial \sigma} + \frac{\partial B_3}{\partial \sigma}\right)\exp(-B_1 - B_2 - B_3) \qquad (23)$$



Each B_i is defined as

$$\left.\begin{aligned} B_1 &= \frac{1}{V_0}\int_V \left(\frac{\sigma}{\sigma_{01}}\right)^{m_1} dx\,dy\,dz \\ B_2 &= \frac{1}{S_0}\int_S \left(\frac{\sigma}{\sigma_{02}}\right)^{m_2} dx\,dz \\ B_3 &= \frac{1}{L_0}\int_L \left(\frac{\sigma}{\sigma_{03}}\right)^{m_3} dx \end{aligned}\right\} \qquad (24)$$

Do the integral with the domain that $\sigma_{xx}(x,y,z)$ is tensile(>0), then we get

$$\left.\begin{aligned} B_1 &= \frac{V_{e0}}{V_0}\left(\frac{\sigma_m}{\sigma_{01}}\right)^{m_1} \\ B_2 &= \frac{S_{e0}}{S_0}\left(\frac{\sigma_m}{\sigma_{02}}\right)^{m_2} \\ B_3 &= \frac{L_{e0}}{L_0}\left(\frac{\sigma_m}{\sigma_{03}}\right)^{m_3} \end{aligned}\right\} \qquad (25)$$

V_{e0}, S_{e0}, L_{e0} are effective volume, surface, length and given by

$$\left.\begin{aligned} V_{e0} &= \frac{bh}{(m_1+1)}\left(\frac{2L_1}{m_1+1}+L_2\right) \\ A_{e0} &= b\left(\frac{2L_1}{m_2+1}+L_2\right) \\ L_{e0} &= \frac{4L_1}{m_3+1}+2L_2 \end{aligned}\right\} \qquad (26)$$

Finally, we obtain competing mode Weibull distribution for 4-point bending test with 3 fracture causes.

$$F(\sigma)=1-\exp\left(-\frac{V_{e0}}{V_0}\left(\frac{\sigma}{\sigma_{01}}\right)^{m_1}-\frac{S_{e0}}{S_0}\left(\frac{\sigma}{\sigma_{02}}\right)^{m_2}-\frac{L_{e0}}{L_0}\left(\frac{\sigma}{\sigma_{03}}\right)^{m_3}\right)$$

Here, σ_m is replaced by σ to coincide to the argument in left hand side.

To apply competing mode Weibull distribution actually, we should determine fracture cause (fracture origin) in every specimen by fractography.

In this case, we have two methods: Weibull plot and maximum likelihood method

For Weibull plot, we should assign the ordered number to strength data.

But in competing mode, ordered number is not integer. And Johnson method or Kaplan/Meer method should be used to give non-integer ordered number.

And then do Weibull plot in each fracture cause, then we get m_i and σ_{oi} for each fracture cause.

For maximum likelihood method, we write down L

$$L = \frac{n!}{\prod_{i=1}^{k}(n_i!)} \prod_{i=1}^{k} L_i \qquad (27)$$

n is total number of specimens

k is number of fracture causes

$$L_i = \left[\prod_{j=1}^{n_i} f_i\left(\sigma_f^{ij}\right)\right]\left[\prod_{p=1,p\neq i}^{k} \prod_{j=1}^{n_p} R_i\left(\sigma_f^{pj}\right)\right] \qquad (28)$$

$$f_i = \frac{\partial B_i}{\partial \sigma}\exp\left(-B_i\right)$$

n_i is total number of specimens fractured by fracture cause i

$$R_i = \exp\left(-B_i\right)$$

σ_f^{ij} is the j-th fracture strength by fracture cause i

To estimate parameters we have two methods, one is direct method.

$$\frac{\partial \ln L}{\partial m_i} = 0 \qquad \left(i = 1,2,\cdots,k\right) \qquad (31)$$

$$\frac{\partial \ln L}{\partial \sigma_{0i}} = 0 \qquad \left(i = 1,2,\cdots,k\right) \qquad (32)$$

But this method does not give good estimation.

Another method is mutiple-step method.

$$\frac{\partial \ln L_i}{\partial m_i} = 0 \qquad (i = 1,2,\cdots,k) \qquad (33)$$

$$\frac{\partial \ln L_i}{\partial \sigma_{0i}} = 0 \qquad (i = 1,2,\cdots,k) \qquad (34)$$

That is, L_i contains only m_i and σ_{oi} only (not m_j and σ_{oj}), so we notice that m_i and σ_{oi} can be estimated by maximizing L_i only.

Do differentiate $\ln L_i$, we obtain the similar non-linear equation as single mode Weibull distribution (but not the same)

This is left for your good exercise.

Merit of Weibull analysis is summarized.

1. You can calculate how much percentage of products is broken when you applied some stress to it.
2. If you give reliability(or fracture probability), you can estimate how much stress you can apply to the products.
3. You can estimate how the reliability of the product is improved by removing one of fracture cause.

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