1 Exercise - Eigenvalue problem

- Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, (d) the algebraic and geometric multiplicity of each eigenvalue.

  (1)
  $$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix},$$

  (2)
  $$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

  (3)
  $$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

- Theorem 4.15,
- Theorem 4.16,
- Theorem 4.17,
- Theorem 4.18,
- Example 4.21,
- Theorem 4.19 and Warning,
- Theorem 4.20 (without proof).
- Let A is a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 2$, respectively, and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

  (a) Find $A^{10}\mathbf{x}$.

  (b) Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty+$)?

- (a) Show that, for any square matrix A, A^T and A have the same characteristic polynomial and hence the same eigenvalues.

  (b) Give an example of a 2×2 matrix A for which A^T and A have different eigenspaces.

- If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

1.1 Solution of exercises

- Compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, (d) the algebraic and geometric multiplicity of each eigenvalue and (e) decide whether A is diagonalizable or not and state why.

  (1)
$$A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix},$$

  (2)
$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

  (3)
$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

  Solution. Follows the same steps as Example 4.19. □

- Let A is a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvaluses $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 2$, respectively, and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

  (a) Find $A^{10}\mathbf{x}$.
  (b) Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty+$)?

  Solution. Follows the same steps as Example 4.21. Notice that we don't need to know the matrix A at all. □

- (a) Show that, for any square matrix A, A^T and A have the same characteristic polynomial and hence the same eigenvalues.
  (b) Give an example of a 2×2 matrix A for which A^T and A have different eigenspaces.

  Solution. As for (a). Since $\det X = \det X^T$ for any square matrix X, we have
$$\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I^T) = \det(A^T - \lambda I).$$

  As for (b). Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. From (a), we see that A has the same eigenvalues as A^T, namely $\lambda_1 = \lambda_2 = 1$, but by a simple calculation we see that the corresponding eigenspaces are different for A and A^T. □

- If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

  Solution. If $A\mathbf{v} = \lambda\mathbf{v}$ then obviously $(A - cI)\mathbf{v} = (\lambda - c)\mathbf{v}$. □

1.2 Extra exercises with solutions

1. read Example 4.4,

2. A proof of Theorem 4.18 (b) and (c).

   Proof. Let $\mathbf{x}$ be an eigenvector of A corresponding to the eigenvalue λ.

   (b) By definition λ is an eigenvalue of A if $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Since A is invertible we have $\lambda \neq 0$ and multiplying the equation by A^{-1} we get

   $$\mathbf{x} = A^{-1}\lambda\mathbf{x}.$$

   $$\therefore A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}.$$

   (c) For positive integer n, by the part (a), λ^n is an eigenvalue of A^n. Since A is invertible, by the part (b), we have λ^{-1} is an eigenvalue of A^{-1}. Using these two information we have

   $$A^{-n}\mathbf{x} = (A^{-1})^n\mathbf{x} = (\lambda^{-1})^n\mathbf{x} = \lambda^{-n}\mathbf{x},$$

   for all positive integers n. In the case $n = 0$ we use the convention $A^0 = I$ and $\lambda^0 = 1$ and hence $A^0\mathbf{x} = I\mathbf{x} = \mathbf{x} = 1\mathbf{x}$ is also satisfied. This completes the proof for all integers n.

   □

3. A proof of Theorem 4.19.

   Proof. Let $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_m$, using Theorem 4.18 (a) we calculate

   $$A^k\mathbf{x} = A^k(c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m) = c_1(A^k\mathbf{v}_1) + \cdots + c_m(A^k\mathbf{v}_m) = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_m\lambda_m^k\mathbf{v}_m.$$

   This concludes the proof for any k positive integer. If A^{-1} existed, we could generalize this results for all integers k. □