2 Lecture - Diagonalization of square matrices

- a note: We consider only real eigenvalues in our theory and calculation! If the characteristic equation of a matrix has complex roots we can not apply the theory to the "only real" part. Please do not apply the theory when there are complex roots.
- definition of similarity,
- Remarks,
- Example 4.22,
- Theorem 4.21,
- Theorem 4.22,
- Remarks,
- Example 4.23,
- definition of diagonalizable matrix,
- Example 4.24,
- Theorem 4.23 (without proof - easy),
- Example 4.25,
- Example 4.26,
- Remarks,
- Theorem 4.24 (without proof - easy)
- Theorem 4.25,
- Lemma 4.26 (without proof),
- Theorem 4.27 (without proof - easy),
- Example 4.28.

2.1 Extra exercises with solutions

1. Let A, B and C be $n \times n$ matrices. Prove that if $A \sim B$ and $B \sim C$, then $A \sim C$.

   Proof. Since $A \sim B$ and $B \sim C$, there exist regular matrices P and Q such that

   $$A = P^{-1}BP,\ B = Q^{-1}CQ.$$

   Hence, $A = P^{-1}(Q^{-1}CQ)P = (QP)^{-1}C(QP)$ and QP is regular. This shows that $A \sim C$. □

2. A proof of (b), (c) and (e) from Theorem 4.22.

   Proof. Let A and B be $n \times n$ matrices with $A \sim B$. Thus there exists invertible matrix P such that $P^{-1}AP = B$ or equivalently $AP = PB$. As for (b), if A is invertible, then A^{-1} exists and

   $$P = A^{-1}AP = A^{-1}PB \Rightarrow I = (P^{-1}A^{-1}P)B.$$

   From the theory in the 1st quarter, B is invertible and $B^{-1} = P^{-1}A^{-1}P$. Since $A \sim B$ implies $B \sim A$, we are finished.

   As for (c), the proof is not difficult, but the chain of reasoning one has to follow is somewhat long. We start by recalling a result from the 1st quarter.

   Lemma 2.1. *If C and P are $n \times n$ matrices such that P is invertible, then*

   $$\mathit{rank}(PC) = \mathit{rank}(C).$$

   Because A is similar to B, there is an invertible matrix P such that $AP = PB$. The lemma gives us that

   $$\text{rank}(B) = \text{rank}(PB).$$

   Of course, $\text{rank}(PB) = \text{rank}(AP)$, and by the Rank Theorem

   $$\text{rank}(AP) = \text{rank}((AP)^T).$$

   Using the lemma once again shows that

   $$\text{rank}((AP)^T) = \text{rank}(P^TA^T) = \text{rank}(A^T).$$

   (Recall that, P^T is invertible if and only if P is invertible). Again applying the Rank Theorem, we conclude that

   $$\text{rank}(B) = \text{rank}(PB) = \text{rank}(AP) = \text{rank}((AP)^T) = \text{rank}(P^TA^T) = \text{rank}(A^T) = \text{rank}(A).$$

   As for (e), it follows immediately from (d), if the characteristic polynomial is the same, the roots (i.e., eigenvalues) are the same.

   □

3. Verify by induction that if $A = PDP^{-1}$ then $A^n = PD^nP^{-1}$ for all $n \in \mathbb{N}$.

Proof. The case $n = 1$ is obvious. Suppose that $A^n = PD^nP^{-1}$ holds. Then

$$\begin{aligned} A^{n+1} &= A \cdot A^n \\ &= (PDP^{-1}) \cdot (PD^nP^{-1}) \\ &= PD^{n+1}P^{-1}. \end{aligned}$$

Which concludes the proof. □

4. Show that A and B are not similar matrices.

a)

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Solution. We check the necessary conditions for $A \sim B$ from Theorem 4.22. We see that $\det A = \det B$, but the characteristic polynomial of each matrix is different.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 1 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 1, \\ |B - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2. \end{aligned}$$

□

b)

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}.$$

Solution. In the same way as above, we see that

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & 1 & 4 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)^2(4-\lambda), \\ |B - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 4-\lambda & 0 \\ 2 & 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)^2. \end{aligned}$$

Hence, A and B are not similar. □

5. Determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$(a) \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad (b) \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution. (a) Let λ denote an eigenvalue of A. By solving

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0,$$

we have the eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

Let us examine the eigenspace for $\lambda = 1$. Since

$$[A - 1 \cdot I|\mathbf{0}] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector when $x_2 = 0$ and $x_3 = 0$. Thus

$$W_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Thus, we see that the algebraic multiplicity and the geometric multiplicity are different. Hence the given matrix A is NOT diagonalizable.

(b) Let λ denote an eigenvalue of A. By solving

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(1+\lambda)(\lambda-2) = 0,$$

we have the eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 2$. Thus we have three distinct eigenvalues of algebraic multiplicity 1, by Theorem 4.20 the corresponding eigenvectors will be linearly independent and since we get 3 they form basis of $\mathbb{R}^3$ and we can use Theorem 4.23 to diagonalize A.

The eigenspaces are, $E_1 = \text{null}(A - 1 \cdot I) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$, $E_{-1} = \text{null}(A + 1 \cdot I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right)$ and $E_2 = \text{null}(A - 2 \cdot I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

So if we define $P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$, A can be diagonalized as

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

thanks to Theorem 4.23.

□

6. Example 4.29.

7. Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}^8.$$

Solution. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and λ be an eigenvalue of A. By solving

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -\lambda^2(\lambda - 2) = 0,$$

we find the eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.

Since

$$[A - 0 \cdot I|\mathbf{0}] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector when $x_1 + x_3 = 0$. Thus

$$E_0 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

In the same way, we find

$$E_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

So if we define $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$, A can be diagonalized as

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\therefore A = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} P^{-1}.$$

Now we calculate

$$
\begin{aligned}
A^8 &= P\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}^8 P^{-1} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2^8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 128 & 0 & 128 \\ 128 & 0 & 128 \\ 128 & 0 & 128 \end{bmatrix}.
\end{aligned}
$$

□