3 Lecture - Orthogonality

3.1 Projections onto vectors in $\mathbb{R}^n$

- recap of dot product, norm, distance and angle in $\mathbb{R}^n$,
- derivation of projection onto a vector in $\mathbb{R}^2$,
- definition of projection onto a vector in $\mathbb{R}^n$,
- remarks.

3.2 Shadows on Wall

Until now we have discussed only projection onto a single vector (or, equivalently, the one-dimensional subspace spanned by that vector). Next, we will see if we can find the analogous formulas for projection onto a plane in $\mathbb{R}^3$.

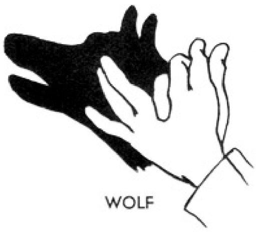
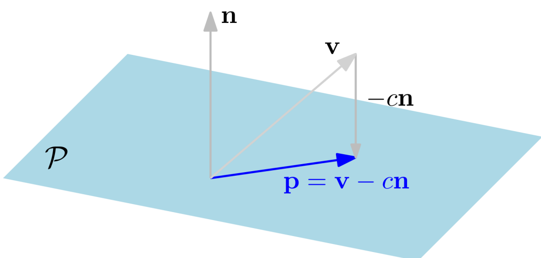


Figure above shows what happens, for example, when parallel light rays create a shadow on a wall. A similar process occurs when a three-dimensional object is displayed on a two-dimensional screen, such as a computer monitor. Later, we will consider these ideas in full generality.

Now let us consider projections onto planes through the origin. We will explore several approaches.

Figure below shows one way to proceed. If $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n}$ and if $\mathbf{v}$ is a vector in $\mathbb{R}^3$, then $\mathbf{p} = \text{proj}_{\mathcal{P}}(\mathbf{v})$ is a vector in $\mathcal{P}$ such that $\mathbf{v} - c\mathbf{n} = \mathbf{p}$ for some scalar c. (Reminder: Normal vector $\mathbf{n}$ of a plane is a vector perpendicular to the plane with norm equal to 1.)



Projection onto a plane

Example 3.1. Using the fact that $\mathbf{n}$ is orthogonal to every vector in $\mathcal{P}$, solve $\mathbf{v} - c\mathbf{n} = \mathbf{p}$ for c to find an expression for $\mathbf{p}$ in terms of $\mathbf{v}$ and $\mathbf{n}$.

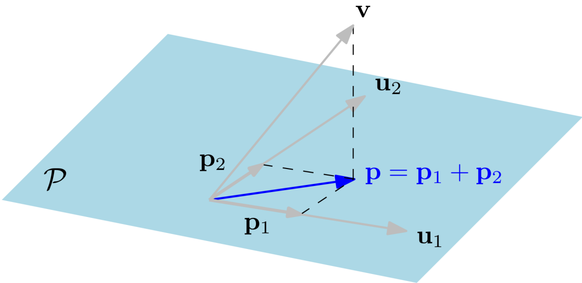
Solution. Since $\mathbf{n}$ is orthogonal to every vector in $\mathcal{P}$ it is in particular orthogonal to $\mathbf{p}$. In algebraic words,

$$\mathbf{p} \cdot \mathbf{n} = 0 \Leftrightarrow (\mathbf{v} - c\mathbf{n}) \cdot \mathbf{n} = 0 \Leftrightarrow c = \mathbf{v} \cdot \mathbf{n}.$$

Hence, $\mathbf{p} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$.

□

Another approach to the problem of finding the projection of a vector onto a plane is suggested by next figure.



Projection onto a plane

We can decompose the projection of $\mathbf{v}$ onto $\mathcal{P}$ into the *sum* of its projections onto the direction vectors for $\mathcal{P}$. This works only if the direction vectors are orthogonal unit vectors. Accordingly, let $\mathbf{u}_1$ and $\mathbf{u}_2$ be direction vectors for $\mathcal{P}$ with the property that

$$\|\mathbf{u}_1\| = \|\mathbf{u}_2\| = 1 \text{ and } \mathbf{u}_1 \cdot \mathbf{u}_2 = 0.$$

By definition of $\text{proj}_{\mathbf{u}}(\mathbf{v})$, the projections of $\mathbf{v}$ onto $\mathbf{u}_1$ and $\mathbf{u}_2$ are

$$\mathbf{p}_1 = (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 \text{ and } \mathbf{p}_2 = (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2,$$

respectively. To show that the $\mathbf{p}_1 + \mathbf{p}_2$ gives us the projection of $\mathbf{v}$ onto $\mathcal{P}$, we need to show that $\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2)$ is orthogonal to $\mathcal{P}$. It is enough to show that $\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2)$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. (Why?)

Example 3.2. Show that $(\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2)) \cdot \mathbf{u}_1 = 0$ and $(\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2)) \cdot \mathbf{u}_2 = 0$.

Solution. Using the orthogonality and unity of $\mathbf{u}_1$ and $\mathbf{u}_2$ we have

$$(\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2))\mathbf{u}_1 = 0 \Rightarrow \mathbf{v} \cdot \mathbf{u}_1 - (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 \cdot \mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2 \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{u}_1 - \mathbf{u}_1 \cdot \mathbf{v} = 0,$$

and

$$(\mathbf{v} - (\mathbf{p}_1 + \mathbf{p}_2))\mathbf{u}_2 = 0 \Rightarrow \mathbf{v} \cdot \mathbf{u}_2 - (\mathbf{u}_1 \cdot \mathbf{v})\mathbf{u}_1 \cdot \mathbf{u}_2 - (\mathbf{u}_2 \cdot \mathbf{v})\mathbf{u}_2 \cdot \mathbf{u}_2 = \mathbf{v} \cdot \mathbf{u}_2 - \mathbf{u}_2 \cdot \mathbf{v} = 0.$$

□

Hence we obtained that

$$\text{proj}_{\mathcal{P}}(\mathbf{v}) = \text{proj}_{\mathbf{u}_1}(\mathbf{v}) + \text{proj}_{\mathbf{u}_2}(\mathbf{v}).$$

3.3 Orthogonality in $\mathbb{R}^n$

- definition of an orthogonal set,
- Theorem 5.1,
- definition of an orthogonal basis,
- Theorem 5.2.
- definition of an orthonormal set and an orthonormal basis,
- remark,
- Theorem 5.3.