2 Homework (solution)

(a) Exercise 1.1 on page 16.

31. $2 + 2 + 2 = 0$ in $\mathbb{Z}_3$.

32. $2 \cdot 2 \cdot 2 = 2$ in $\mathbb{Z}_3$.

33. $2(2 + 1 + 2) = 1$ in $\mathbb{Z}_3$.

34. $3(3 + 3 + 2) = 0$ in $\mathbb{Z}_4$.

35. $2 \cdot 3 \cdot 2 = 0$ in $\mathbb{Z}_4$.

36. $3 + 1 + 2 + 3 = 1$ in $\mathbb{Z}_4$.

44. $x = 4$ in $\mathbb{Z}_5$.

45. $x = 2$ in $\mathbb{Z}_6$.

49. There is no solution of $2x = 1$ in $\mathbb{Z}_4$. (Try any value from 0 to 3)

56. (a) For all, just test all values.
    (b) For all, just test all values.
    (c) From previous results we suspect that the answer should be for all. Lets try to prove it. If for each a and m there exists $y \in \mathbb{Z}_m$ such that $a + y = 0$ then $x + a = b$ has the solution $x = b + y$ for all a, b and m. So let us show that this is true. For $a = 0$ it is obviously true, we can take $y = 0$. Now let $a > 0$ then $m - a \leq m - 1$ and obviously $m - a \geq 1$ and hence $y = m - a \in \mathbb{Z}_m$. Proved.

57. When $a = 0$ there is never a solution and when $a = 1$ there is always the solution $x = 1$ irrespective of m.

    (a) There is a solution for $a = 1, 2, 3, 4$.
    (b) There is a solution only for $a = 1, 5$.
    (c) There is a solution if and only if $GCD(a, m) = 1$, in other words if the greatest common divisor of a and m is equal to 1. Here is a proof.

        We have to show two implications. First is from left to right, if there is a solution then $GCD(a, m) = 1$. To do so we prove an equivalent implication, namely, if $GCD(a, m) = p > 1$ then there is no solution.

        If p divides both a and m, then p also divides ax but then it does not divide $ax - 1$, hence $ax - 1$ can never be 0 modulo m.

        Second we prove the implication from right to left, i.e., if $GCD(a, m) = 1$ then there is a solution. We do that by finding a particular solution.

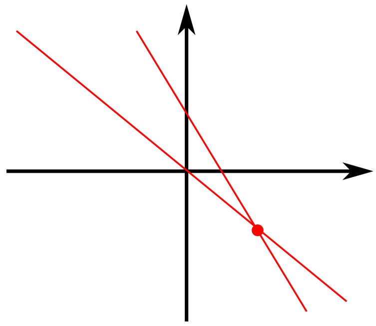
        Let $GCD(a, m) = 1$ and consider $a, a^2, a^3, ...$ These fall in finitely many remainder classes, thus there exist $k < l$ so that $a^k = a^l \bmod m$. That is, $A = a^k(a^(l - k) - 1)$ is divisible by m. If p is a prime factor of m so that p^s divides m but $p^(s + 1)$ does not, then p^s divides A but because a is not divisible by p, this implies that p^s divides $a^(l - k) - 1$. Since this holds for all p, we have that m divides $a^(l - k) - 1$. That means that $x = a^(l - k - 1)$ is a solution to $ax = 1 \bmod m$.

(b) Exercise 2.1 on page 69:

8. The given system is equivalent to $x + y = 1, (x - y \neq 0)$.

9. The given system is equivalent to $x + y = 4, (xy \neq 0)$.

15. From the figure, there is a unique answer $x = 3.y = -3$.



25. $x = 2, y = -7, z = -32$.

39. (a) Computing the augmented matrix in the inverse direction, we have $\left[\begin{array}{cc|c} 1 & 0 & t \\ 0 & 1 & 3-2t \end{array}\right] \to$ $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 3-2t \end{array}\right]$. Thus the desired equation is $2x + y = 3$.

(b) Substituting $y = s$ into $2x + y = 3$, we obtain $x = \dfrac{3-s}{2}$.

$$\therefore x = \frac{3-s}{2}, y = s.$$

42. For $x, y \in \mathbb{R}$ let $u = x^2$ and $v = y^2$. The the given system of equations is changed to the system $\begin{cases} u + 2v = 6 \\ u - v = 3 \end{cases}$ for all $u, v \geq 0$. This system of linear equations has the solution $u = 4, v = 1$. This is equivalent to $x^2 = 4, y^2 = 1$.

$$\therefore (x, y) = (2, 1), (2, -1), (-2, 1), (-2, -1).$$

43. For all $x \in (-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $k \in \mathbb{Z}$, $y, z \in \mathbb{R}$ let $u = \tan x$, $v = \sin y$, $w = \cos z$. Then the given system of equations is changed to the system $\begin{cases} u - 2v = 2 \\ u - v + w = 2 \\ v - w = -1 \end{cases}$ for all $u \in \mathbb{R}$, $v, w \in [-1, 1]$. This system of linear equations has the solution $u = 1$, $v = -\frac{1}{2}$, $w = \frac{1}{2}$.

$$\therefore x = \frac{\pi}{4} + k\pi, \quad y = \frac{\pi}{6} + 2k\pi, \quad z = \frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}.$$

44. For all $a, b \in \mathbb{R}$ let $a = \log_2 x$ and $b = \log_3 y$. The the given system of equations is changed to the system $\begin{cases} -x + 2y = 1 \\ 3x - 4y = 1 \end{cases}$ for $x, y > 0$. This system of linear equations has the solution $x = -\frac{4}{5}$, $y = \frac{1}{10}$ and hence there is no solution for the original system since $x = -\frac{4}{5} < 0$.

(c) Exercise 2.2 on page 85:

1. no.

2. yes, yes.

3. yes, yes.

4. yes, no.

5. no.

6. yes, no.

7. no.

8. no.

12. (a) $\begin{bmatrix} 2 & -4 & 2 & 6 \\ 3 & -6 & 2 & 6 \end{bmatrix} \overset{R_2-\frac{3}{2}R_1}{\longrightarrow} \begin{bmatrix} 2 & -4 & 2 & 6 \\ 0 & 0 & -1 & -3 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & -4 & 2 & 6 \\ 3 & -6 & 2 & 6 \end{bmatrix} \overset{R_2-\frac{3}{2}R_1}{\longrightarrow} \begin{bmatrix} 2 & -4 & 2 & 6 \\ 0 & 0 & -1 & -3 \end{bmatrix} \overset{\frac{1}{2}R_1+R_2,(-1)R_2}{\longrightarrow} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

13. (a) $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \overset{3R_2-2R_1,3R_3-4R_1}{\longrightarrow} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \overset{R_3+R_2}{\longrightarrow} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix} \overset{\text{as above}}{\longrightarrow} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1+2R_2}{\longrightarrow} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{\frac{1}{3}R_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

17. Since $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the given matrices are row equivalent. The sequence (for example) is $R_2 - 3R_1$, $R_1 + R_2$, $-\frac{1}{2}R_2$, $-R_2$, $R_3 + 3R_1$ and $R_1 \leftrightarrow R_2$.

20.

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \overset{R_2+R_1}{\longrightarrow} \begin{bmatrix} r_1 \\ r_2 + r_1 \end{bmatrix} \overset{R_1-R_2}{\longrightarrow} \begin{bmatrix} -r_2 \\ r_2 + r_1 \end{bmatrix} \overset{R_2+R_1}{\longrightarrow} \begin{bmatrix} -r_2 \\ r_1 \end{bmatrix} \overset{-R_1}{\longrightarrow} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}$$

Hence the net effect is just interchanging the two rows.

21. It can be achieved only by a sequential application of elementary row operations not by a single operation! Using elementary row operations we have

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \overset{3R_2}{\longrightarrow} \begin{bmatrix} 3 & 1 \\ 6 & 12 \end{bmatrix} \overset{R_2-2R_1}{\longrightarrow} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}.$$

23. (1). 3

(2). 0

(3). 2

(4). 2

(5). 2

(6). 3

(7). 3

(8). 3

24.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&*\\0&1&*\\0&0&0\end{bmatrix}, \begin{bmatrix}1&*&0\\0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

$$\begin{bmatrix}1&*&*\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&*\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

31.

$$\left[\begin{array}{ccccc|c}\frac{1}{2}&1&-1&-6&0&2\\\frac{1}{6}&\frac{1}{2}&0&-3&1&-1\\\frac{1}{3}&0&-2&0&-4&8\end{array}\right] \to \left[\begin{array}{ccccc|c}1&0&-6&0&-12&24\\0&1&2&-6&6&-10\\0&0&0&0&0&0\end{array}\right].$$

The solution set is $\left\{ s\begin{bmatrix}6\\-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\6\\0\\1\\0\end{bmatrix} + u\begin{bmatrix}12\\-6\\0\\0\\1\end{bmatrix} + \begin{bmatrix}24\\10\\0\\0\\0\end{bmatrix} \mid s,t,u \in \mathbb{R} \right\}.$

32.

$$\left[\begin{array}{ccc|c}\sqrt{2}&1&2&1\\0&\sqrt{2}&-3&-\sqrt{2}\\0&-1&\sqrt{2}&1\end{array}\right] \to \left[\begin{array}{ccc|c}\sqrt{2}&0&0&2\\0&1&0&-1\\0&0&1&0\end{array}\right].$$

The unique solution is $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\-1\\1\end{bmatrix}.$

33.

$$\left[\begin{array}{cccc|c}1&1&2&1&1\\1&-1&-1&1&0\\0&1&1&0&-1\\1&1&0&1&2\end{array}\right] \to \left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{array}\right].$$

The equation has no solution.

35. Unique solution - It is in row echelon form with all leading 1's.

36. No solution - coefficients of 3rd row are multiple of 2nd row, but constant terms are not.

37. Infinitely many solutions - homogeneous system - thus it is consistent and there is more variables than equations.

38. Infinitely many solutions - 3rd row is a linear combination of 1st and 2 row so we can forget about it. Since it is less equations than variables the system has either infinitely many solutions or no solution. (Do you understand why?) If a system has no solution then its row echelon form contains a row consisting only of zeros at the "coefficient part" and a nonzero constant term on the right hand side. (Do you understand why?) We can easily see that our matrix does not have such a row echelon form. (Do you understand why?) Hence the system has infinitely many solutions.

41. $\left[\begin{array}{cc|c}1&k&1\\k&1&1\end{array}\right] \to \left[\begin{array}{cc|c}1&k&1\\0&1-k^2&1-k\end{array}\right]$. If $k=1$, the augmented matrix is $\left[\begin{array}{cc|c}1&1&1\\0&0&0\end{array}\right]$. Thus it has infinitely many solutions.

If $k=-1$, the augmented matrix is $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$. Thus it has no solution.

If $k \neq \pm 1$, the augmented matrix is $\left[\begin{array}{cc|c} 1 & 0 & \dfrac{k}{k+1} \\ 0 & 1 & \dfrac{1}{k+1} \end{array}\right]$. Thus it has a unique solution.

42. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 4 & -1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & (k+3)(k-2) \end{array}\right]$.

If $k \neq -3, 2$, there is no solution.

If $k=-3$ the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & \frac{11}{3} \\ 0 & 1 & -\frac{2}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus it has infinitely many solutions.

If $k=2$, the augmented matrix is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 2 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus it has infinitely many solutions.

44. The system $x+y=1$, $2x+2y=2$ satisfies $m=n=2$ and has infinitely many solutions. The system $x+y=1$, $2x+2y=2$, $3x+3y=3$ satisfies $m=3>2=n$ and has infinitely many solutions.

Similarly, $x+y=1$, $x-y=0$ satisfies $m=n=2$ and has a unique solution and $x+y=1$, $x-y=0$, $2x-2y=0$ satisfies $m=3>2=n$ has a unique solution.

54. $\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 1 \\ 0 & 0 & 3 \end{array}\right]$. There is no solution.

55. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$. Thus $x=1, y=0, z=0$.

56. $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Thus the solution set is

$$\left\{ s\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} \mid s \in \mathbb{R} \right\}.$$

60. $\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 3 & 2 \end{array}\right]$ in $\mathbb{Z}_6$. Using Gaussian elimination we get by R_2+R_1 that $\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & 0 \end{array}\right]$ and hence the solutions $[x, y]$ are

$$[2,0], [5,0], [2,2], [5,2], [2,4], [5,4].$$

We have one free variable and hence we expect 6 solutions, one for each choice of $y \in \{0, \ldots, 5\}$. We got 6 solutions however for $y=0,2,4$ we got TWO different solutions and for $y=1,3,5$ we have no solution. That is not a behavior we expect from a linear equation. Hence working with $\mathbb{Z}_p$ where p is prime is necessary.