1 Homework (solution)

(a) Algebraical proof of statements (b) to (h) in Theorem 1.1.

Solution. In the proof we are using properties of the arithmetic of real numbers.

(a) in textbook

(b) in textbook

(c)

$$\mathbf{u}+\mathbf{0}=\begin{bmatrix} u_1+0 \\ \vdots \\ u_n+0 \end{bmatrix}=\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}=\mathbf{u}.$$

(d) Using (f) and (h) which we prove independently we have

$$\mathbf{u}+(-\mathbf{u})=1\mathbf{u}+(-1)\mathbf{u}=(1-1)\mathbf{u}=0\mathbf{u}=\mathbf{0}.$$

(e)

$$c(\mathbf{u}+\mathbf{v})=\begin{bmatrix} c(u_1+v_1) \\ \vdots \\ c(u_n+v_n) \end{bmatrix}=\begin{bmatrix} cu_1+cv_1 \\ \vdots \\ cu_n+cv_n \end{bmatrix}=c\mathbf{u}+c\mathbf{v}.$$

(f)

$$(c+d)\mathbf{u}=\begin{bmatrix} (c+d)u_1 \\ \vdots \\ (c+d)u_n \end{bmatrix}=\begin{bmatrix} cu_1+du_1 \\ \vdots \\ cu_n+du_n \end{bmatrix}=c\mathbf{u}+d\mathbf{u}.$$

(g)

$$c(d\mathbf{u})=\begin{bmatrix} c(du_1) \\ \vdots \\ c(du_n) \end{bmatrix}=\begin{bmatrix} cdu_1 \\ \vdots \\ cdu_n \end{bmatrix}=(cd)\mathbf{u}.$$

(h)

$$1\mathbf{u}=\begin{bmatrix} 1u_1 \\ \vdots \\ 1u_n \end{bmatrix}=\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}=\mathbf{u}.$$

□

(b) Algebraical proof of statements (b) to (d) in Theorem 1.2.

Solution. (a) in textbook

(b)

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=u_1(v_1+w_1)+\cdots+u_n(v_n+w_n)=u_1v_1+\cdots+u_nv_n+u_1w_1+\cdots+u_nw_n=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}.$$

(c) in textbook

(d) Since $\mathbf{u}\cdot\mathbf{u} = \sum_{i=1}^{n} u_i^2$ and $a^2 \geq 0$ for all $a \in \mathbb{R}$ we have

$$\mathbf{u}\cdot\mathbf{u} \geq 0.$$

Next we prove the implication from left to right. If $\mathbf{u}\cdot\mathbf{u} = 0$ then $\sum_{i=1}^{n} u_i^2 = 0$ which is true only if $u_i = 0$ for all $i = 1, \dots, n$. Hence,

$$\mathbf{u} = \mathbf{0}.$$

The implication form right to left is simply

$$\mathbf{u} = \mathbf{0} \Rightarrow \mathbf{u}\cdot\mathbf{u} = \sum_{i=1}^{n} 0^2 = 0.$$

□

(c) Exercise 1.1 on page 16:

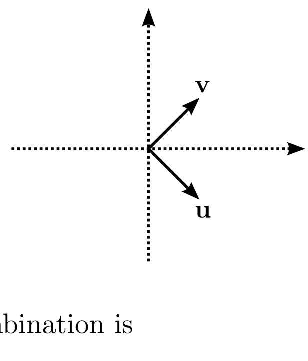
12. $2\mathbf{c} - 3\mathbf{b} - \mathbf{d} = [-6, -9, 1]$.

17.

$$\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a})$$
$$\mathbf{x} - \mathbf{a} = 2\mathbf{x} - 4\mathbf{a}\backslash - \mathbf{x}$$
$$-\mathbf{a} = \mathbf{x} - 4\mathbf{a}\backslash + 4\mathbf{a}$$
$$3\mathbf{a} = \mathbf{x}$$
$$\mathbf{x} = 3\mathbf{a}$$

21. From the following diagram, we have

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}(\mathbf{u}+\mathbf{v}), \ \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}(-\mathbf{u}+\mathbf{v}).$$



Thus a desired linear combination is

$$\mathbf{w} = \begin{bmatrix}2\\6\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 6\begin{bmatrix}0\\1\end{bmatrix} = 2\cdot\frac{1}{2}(\mathbf{u}+\mathbf{v}) + 6\cdot\frac{1}{2}(-\mathbf{u}+\mathbf{v}) = -2\mathbf{u} + 4\mathbf{v}.$$

25. $\mathbf{u}+\mathbf{v} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{u}\cdot\mathbf{v} = 1$.

26. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}\cdot\mathbf{v} = 0$.

27. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}\cdot\mathbf{v} = 1$.

28. $\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}\cdot\mathbf{v} = 0$.

(d) Exercise 1.2 on page 29:

2. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 6 \end{bmatrix} = 0$. They are orthogonal.

3. $\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 11$.

8. $\|\mathbf{u}\| = \left\| \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\| = \sqrt{13}$. The unit vector is $\dfrac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

9. $\|\mathbf{u}\| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\| = \sqrt{14}$. The unit vector is $\dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

14. $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{130}$.

15. $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{6}$.

17. (a) $\mathbf{u}\cdot\mathbf{v}$ is a scalar but $\|\cdot\|$ is defined for a vector.

(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar and $\mathbf{w}$ is a vector.

(c) $\mathbf{u}$ is a vector and $(\mathbf{v}\cdot\mathbf{w})$ is a scalar. Dot product between vectors and scalars is not defined.

(d) c is a scalar and $\mathbf{u}+\mathbf{v}$ is a vector. Dot product between scalars and vectors is not defined.

24. $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\dfrac{\sqrt{2}}{10}$. Therefore $\theta = \arccos\left(-\dfrac{\sqrt{2}}{10}\right)$.

25. $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\dfrac{1}{2}$. Therefore $\theta = \frac{\pi}{3}$.

52. Since

(a)

$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + 2(\mathbf{u}\cdot\mathbf{v} - \|\mathbf{u}\|\|\mathbf{v}\|) \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2 + 2(\mathbf{u}\cdot\mathbf{v} - \|\mathbf{u}\|\|\mathbf{v}\|), \end{aligned}$$

We see that

$$\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|,$$

holds if $\|\mathbf{u}\|\|\mathbf{v}\| = \mathbf{u}\cdot\mathbf{v}$. In other words, $\mathbf{u}$ and $\mathbf{v}$ has to satisfy $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = 1$, which in $\mathbb{R}^2$ or $\mathbb{R}^2$ means that $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

(b) Analogously to previous calculation we find out that $\|\mathbf{u}\|\|\mathbf{v}\| = -\mathbf{u}\cdot\mathbf{v}$ has to be satisfied and hence $\mathbf{u}$ and $\mathbf{v}$ must be oriented in oposite direction to each other.

55. $d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = d(\mathbf{v},\mathbf{u})$.

59. By the triangle inequality $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$. Let us change $\mathbf{u}$ to $\mathbf{u}-\mathbf{v}$. Then we obtain

$$\|\mathbf{u}\| \le \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|,$$

and hence

$$\|\mathbf{u}\| - \|\mathbf{v}\| \le \|\mathbf{u}-\mathbf{v}\|.$$

63.

$$\begin{aligned}
(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= \|\mathbf{u}\|^2 - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \|\mathbf{v}\|^2\\
&= \|\mathbf{u}\|^2 - \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v} - \|\mathbf{v}\|^2\\
&= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.
\end{aligned}$$

66.

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}\\
&= 4+3+2\\
&= 9.
\end{aligned}$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| = 3.$$