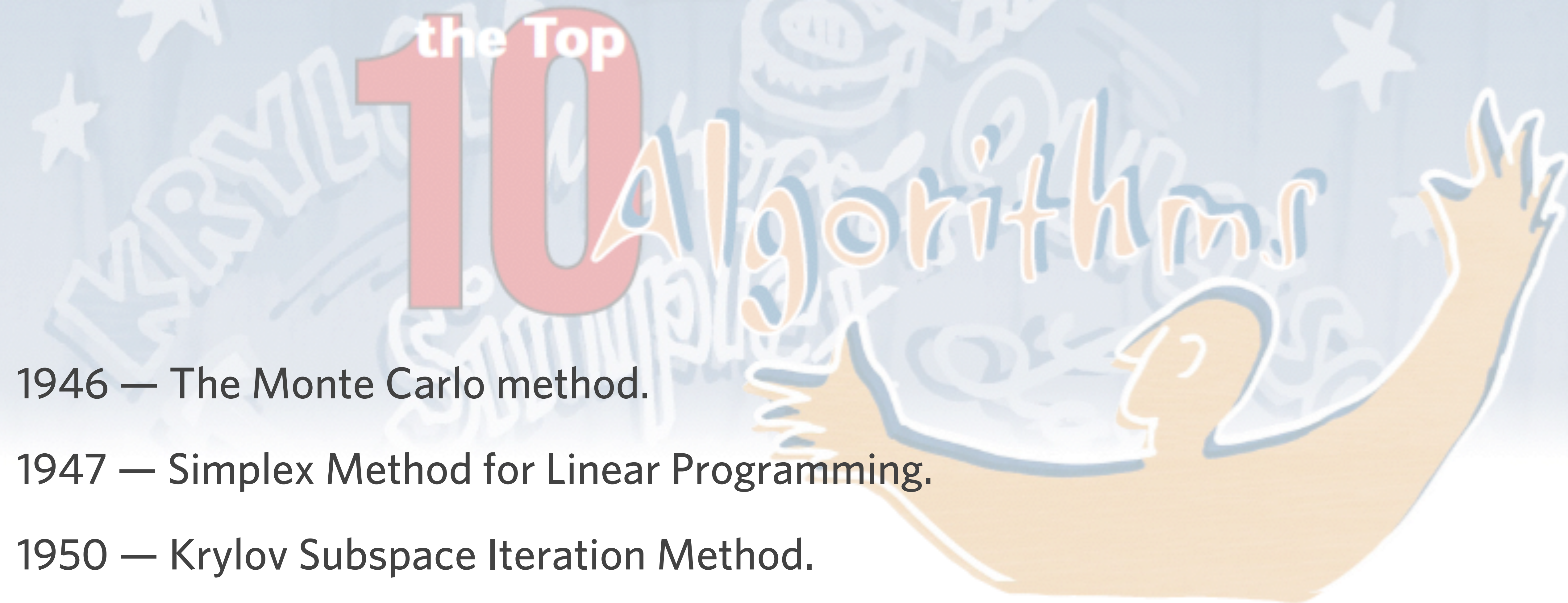
05/09	Class 9	Dense direct solvers	Understand the principle of LU decomposition and the optimization and parallelization techniques that lead to the LINPACK benchmark.
05/12	Class 10	Dense eigensolvers	Determine eigenvalues and eigenvectors and understand the fast algorithms for diagonalization and orthonormalization.
05/16	Class 11	Sparse direct solvers	Understand reordering in AMD and nested dissection, and fast algorithms such as skyline and multifrontal methods.
05/19	Class 12	Sparse iterative solvers	Understand the notion of positive definiteness, condition number, and the difference between Jacobi, CG, and GMRES.
05/23	Class 13	Preconditioners	Understand how preconditioning affects the condition number and spectral radius, and how that affects the CG method.
05/26	Class 14	Multigrid methods	Understand the role of smoothers, restriction, and prolongation in the V-cycle.
05/30	Class 15	Fast multipole methods, H-matrices	Understand the concept of multipole expansion and low-rank approximation, and the role of the tree structure.

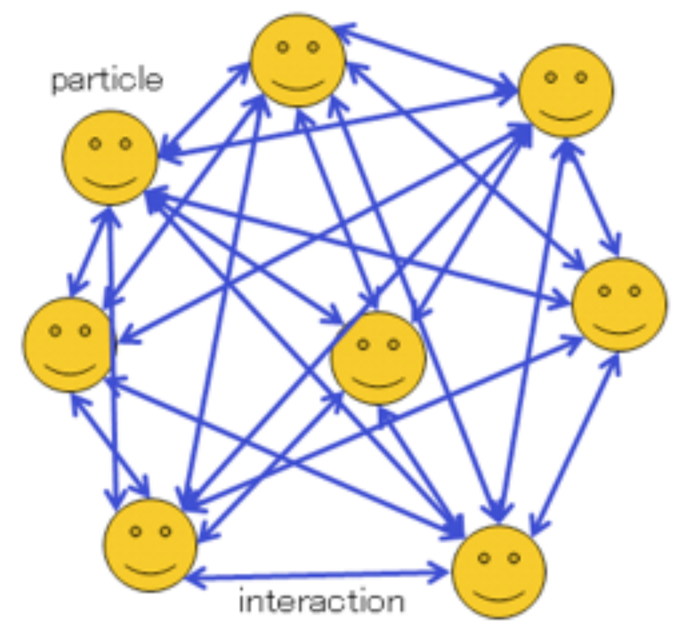


- 1946 — The Monte Carlo method.
- 1947 — Simplex Method for Linear Programming.
- 1950 — Krylov Subspace Iteration Method.
- 1951 — The Decompositional Approach to Matrix Computations.
- 1957 — The Fortran Compiler.
- 1959 — QR Algorithm for Computing Eigenvalues.
- 1962 — Quicksort Algorithms for Sorting.
- 1965 — Fast Fourier Transform.
- 1977 — Integer Relation Detection.
- 1987 — Fast Multipole Method

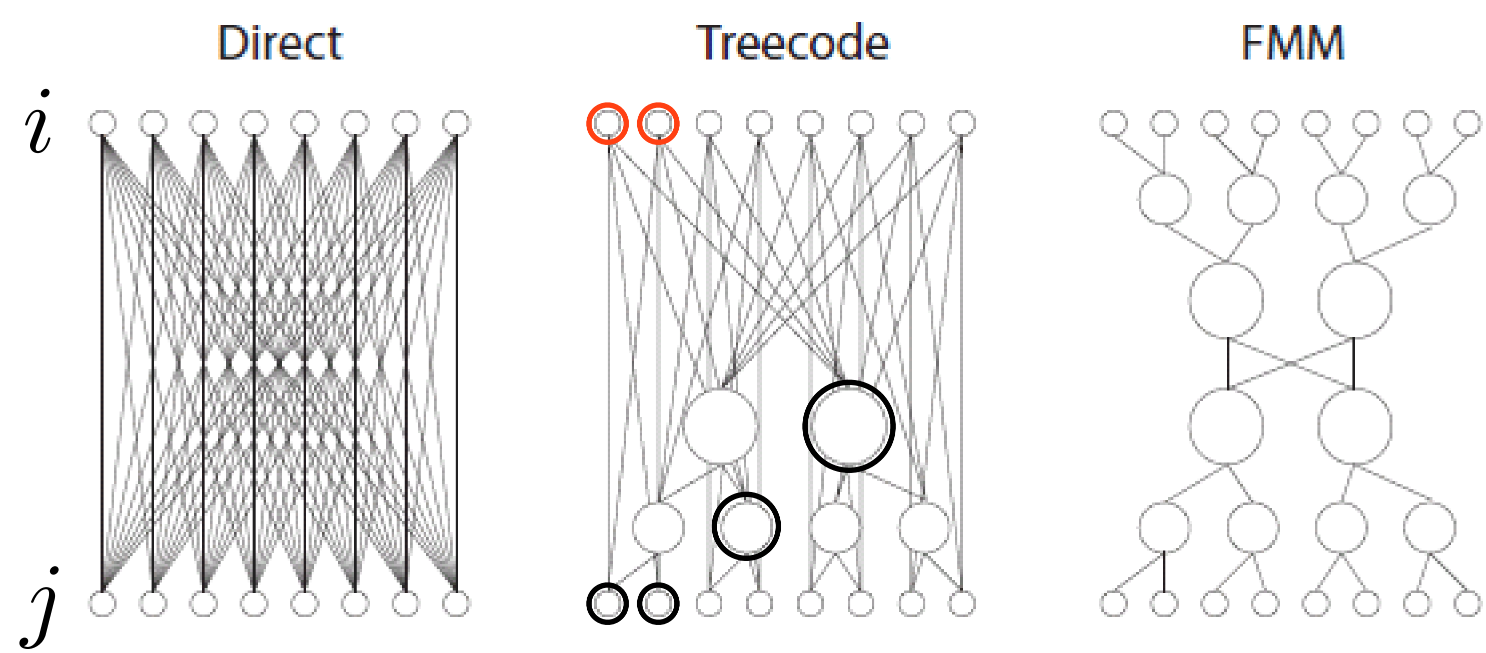
Dongarra& Sullivan, IEEE Comput. Sci. Eng., Vol. 2(1):22-- 23 (2000)

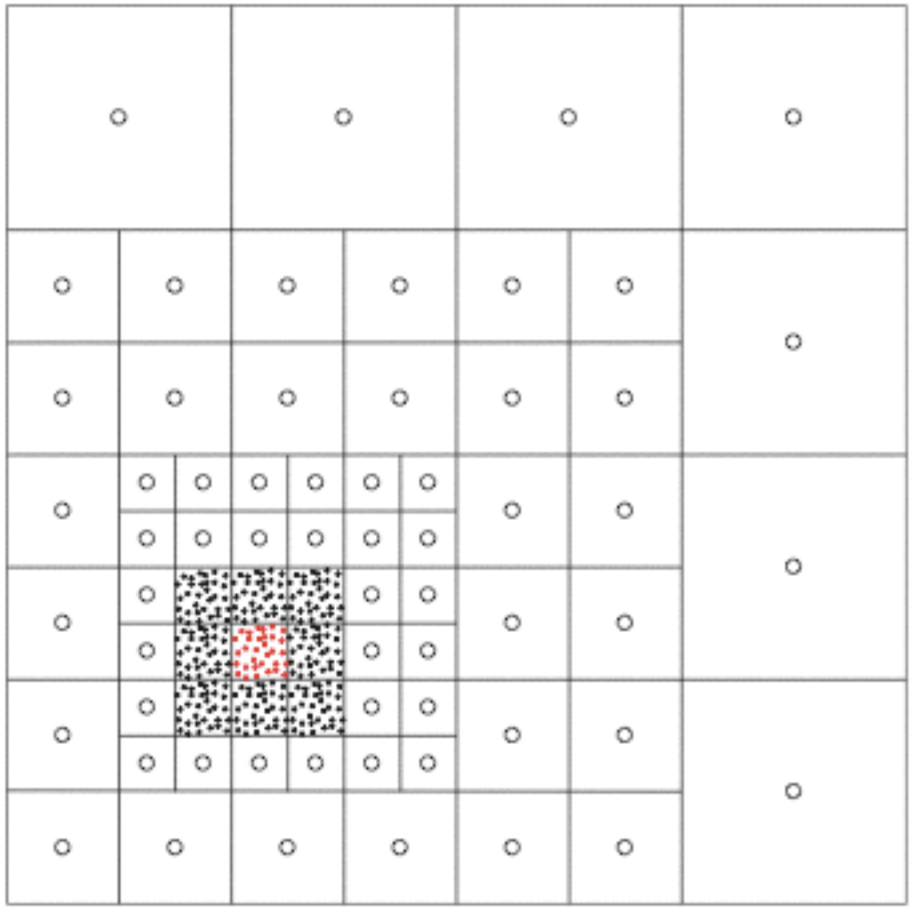
Hierarchical N-body methods

Particles interact with each other
Stars, Galaxies, Atoms, etc.



Computational cost
Direct sum - $O(N^2)$
Treecode - $O(N \log N)$
Fast Multipole Method - $O(N)$





FMM as a Matrix

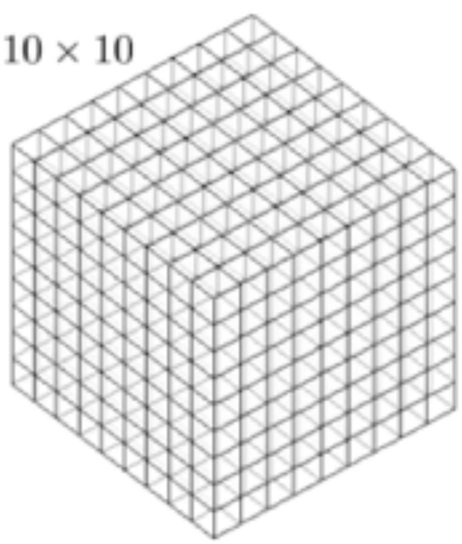
$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y}) f(\mathbf{y})$$

$$\mathbf{u} = A^{-1}\mathbf{b} = G\mathbf{b}$$

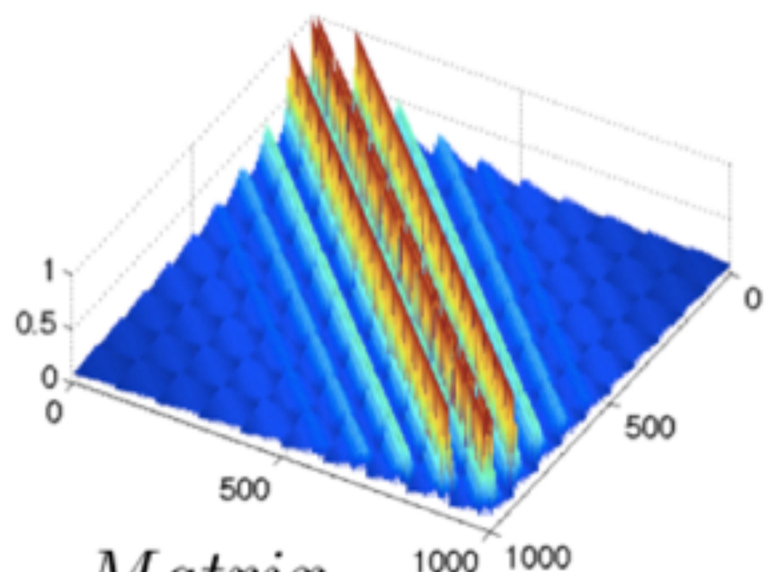
Laplace

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}$$

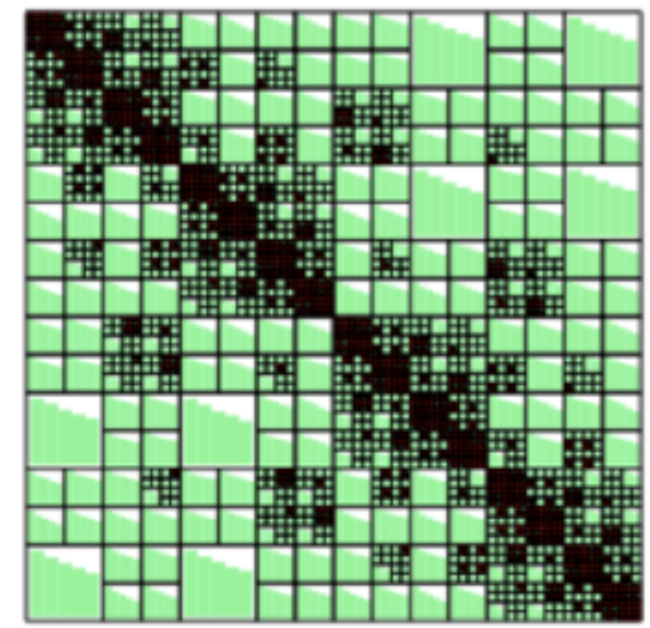
H-matrix

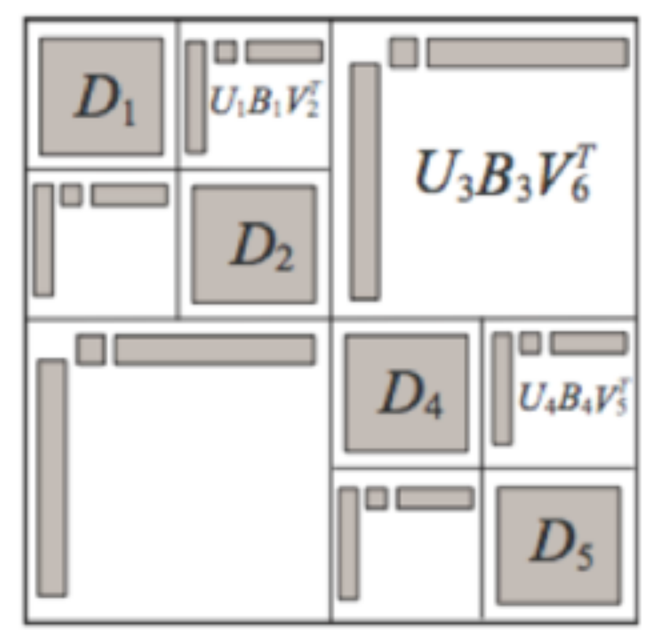


Geometry

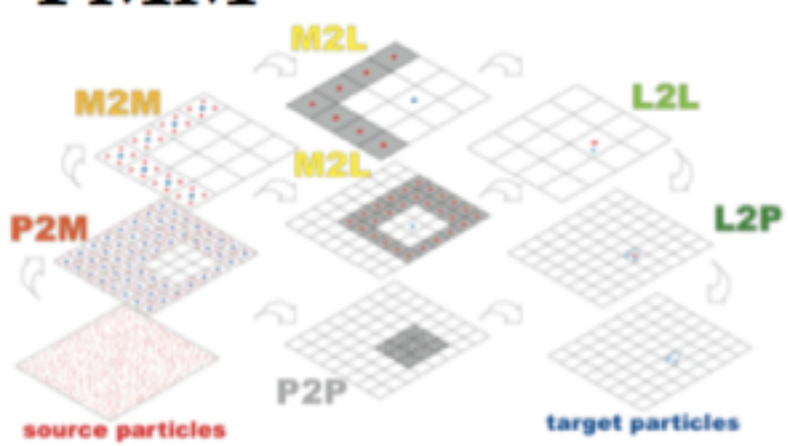


Matrix





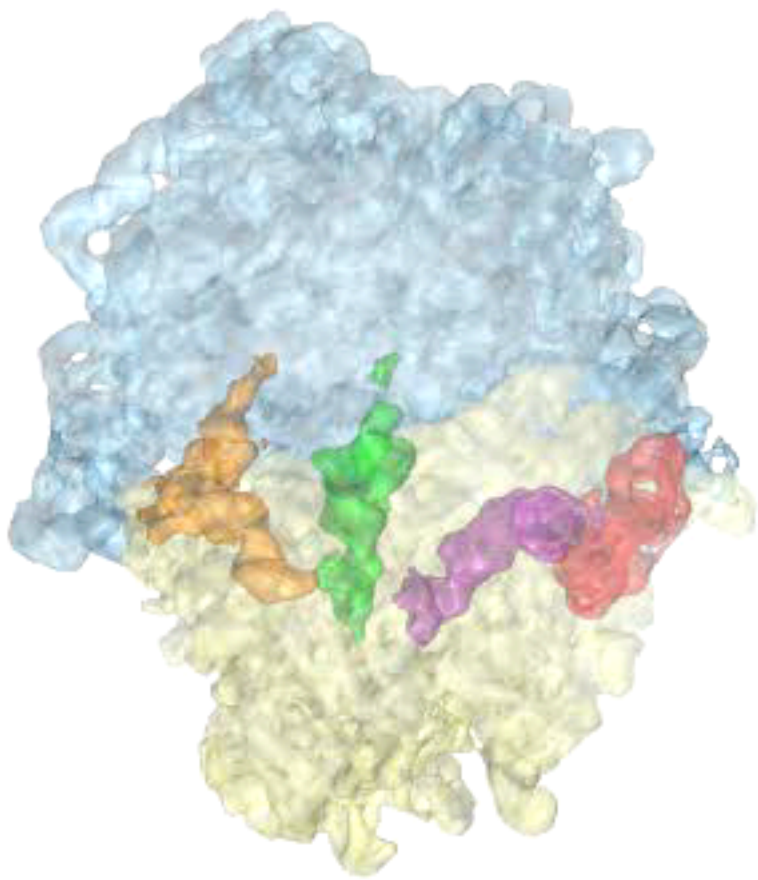
FMM

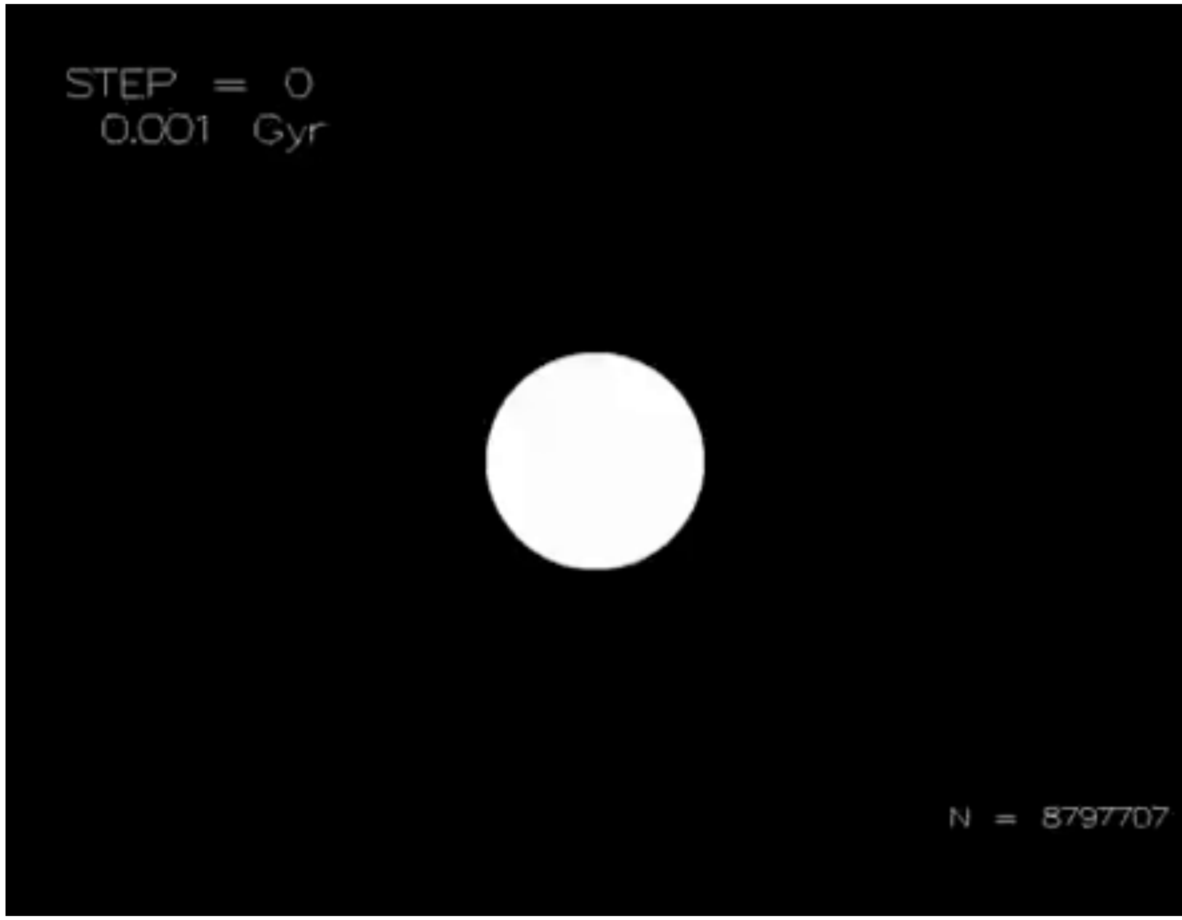


$$G(\mathbf{x}_{ij}) = \sum_{\mathrm{n}=0}^{p} \frac{1}{\mathbf{n}!}(\mathbf{x}_{i\Lambda} + \mathbf{x}_{Mj})^{\mathrm{n}} \nabla^{(\mathrm{n})} G(\mathbf{x}_{\Lambda M})$$

$$= \underbrace{\sum_{\mathrm{k}=0}^{p} \frac{1}{\mathbf{k}!}\mathbf{x}_{i\Lambda}^{\mathrm{k}} \sum_{\mathrm{n}=0}^{p-\mathrm{k}} \nabla^{(\mathrm{n}+\mathrm{k})} G(\mathbf{x}_{\Lambda M}) \underbrace{\frac{1}{\mathbf{n}!}\mathbf{x}_{Mj}^{\mathrm{n}}}_{\mathrm{M}}}_{\mathrm{L}}$$

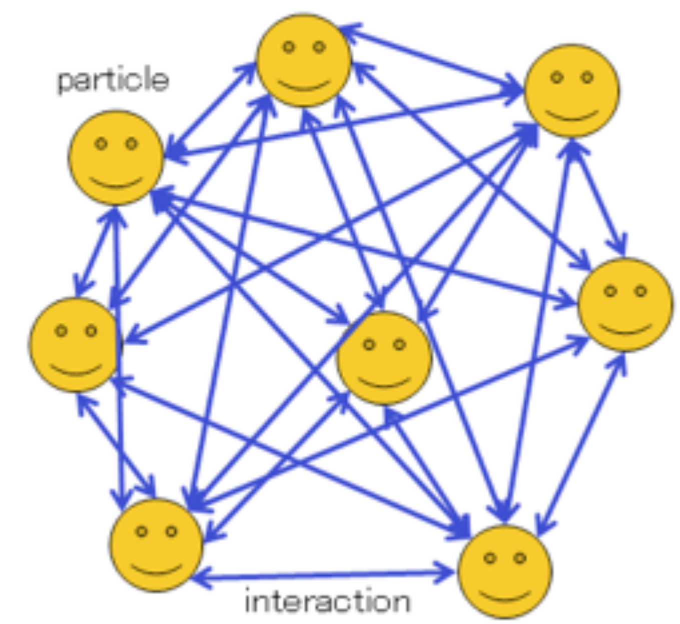
N-body problems



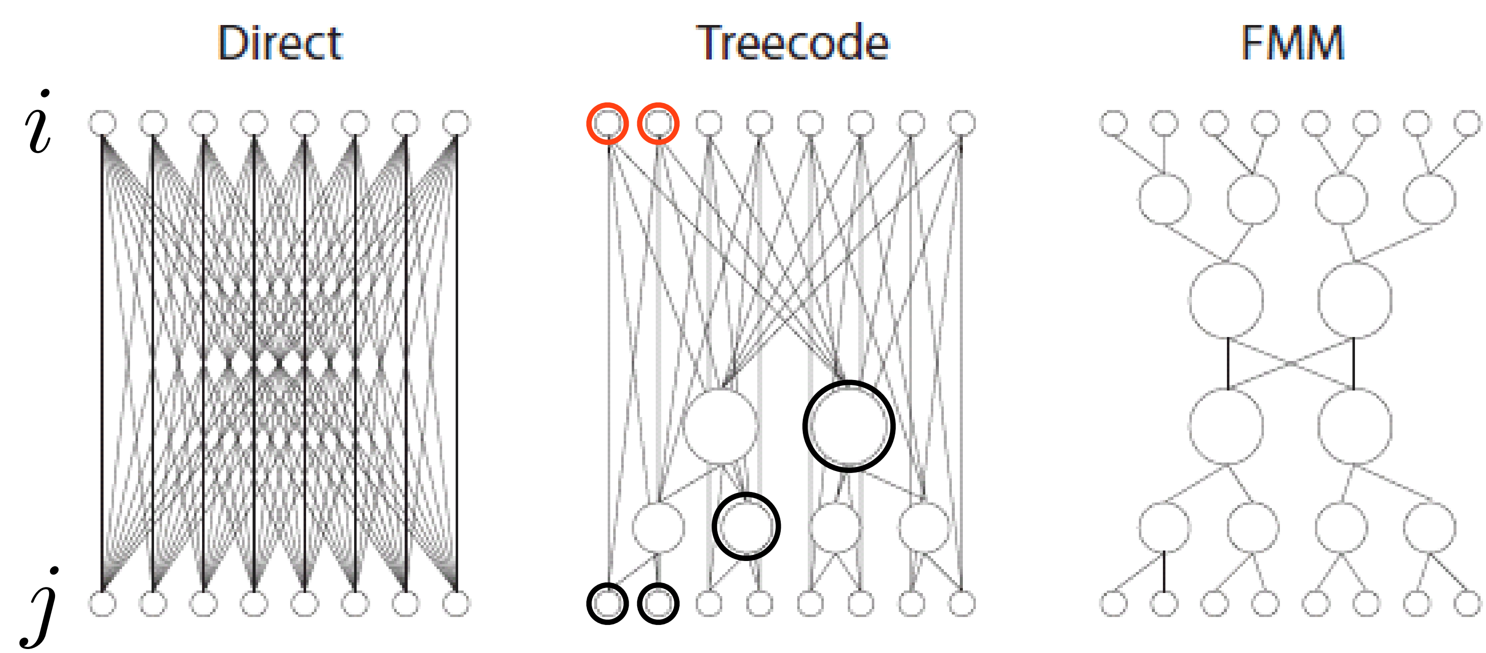


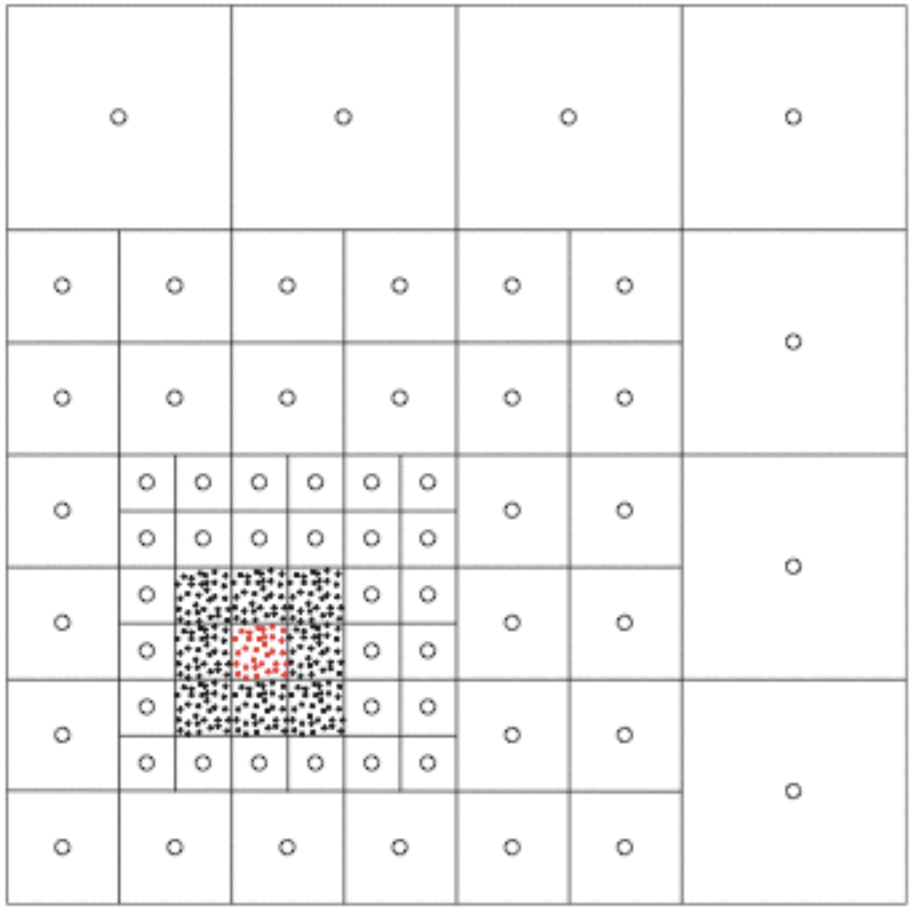
Hierarchical N-body methods

Particles interact with each other
Stars, Galaxies, Atoms, etc.

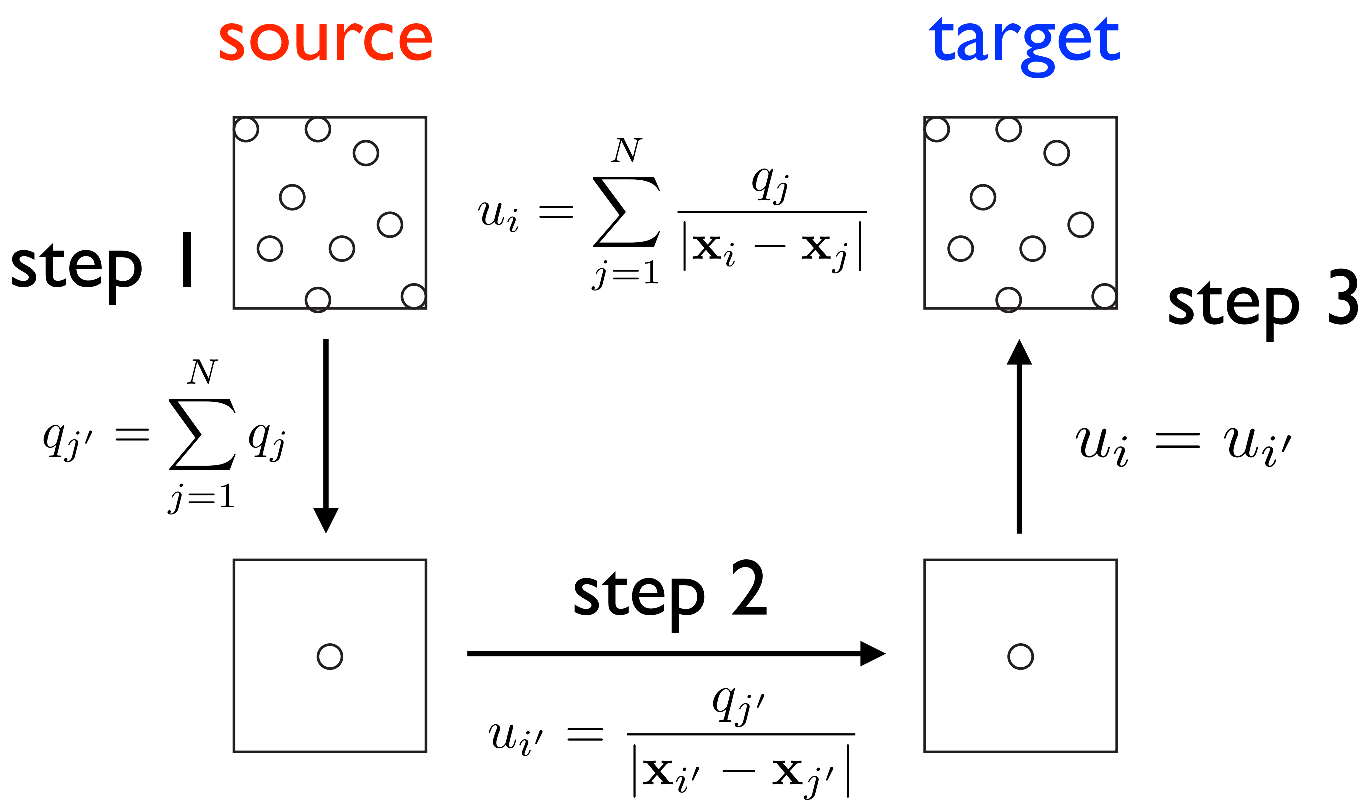


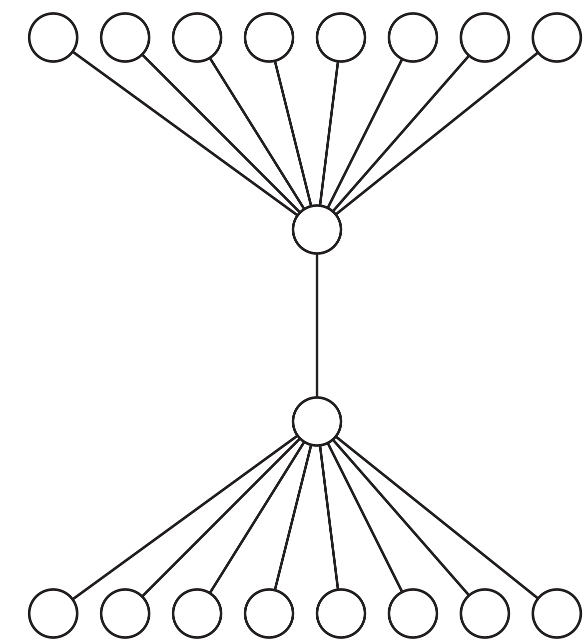
Computational cost
Direct sum - $O(N^2)$
Treecode - $O(N \log N)$
Fast Multipole Method - $O(N)$





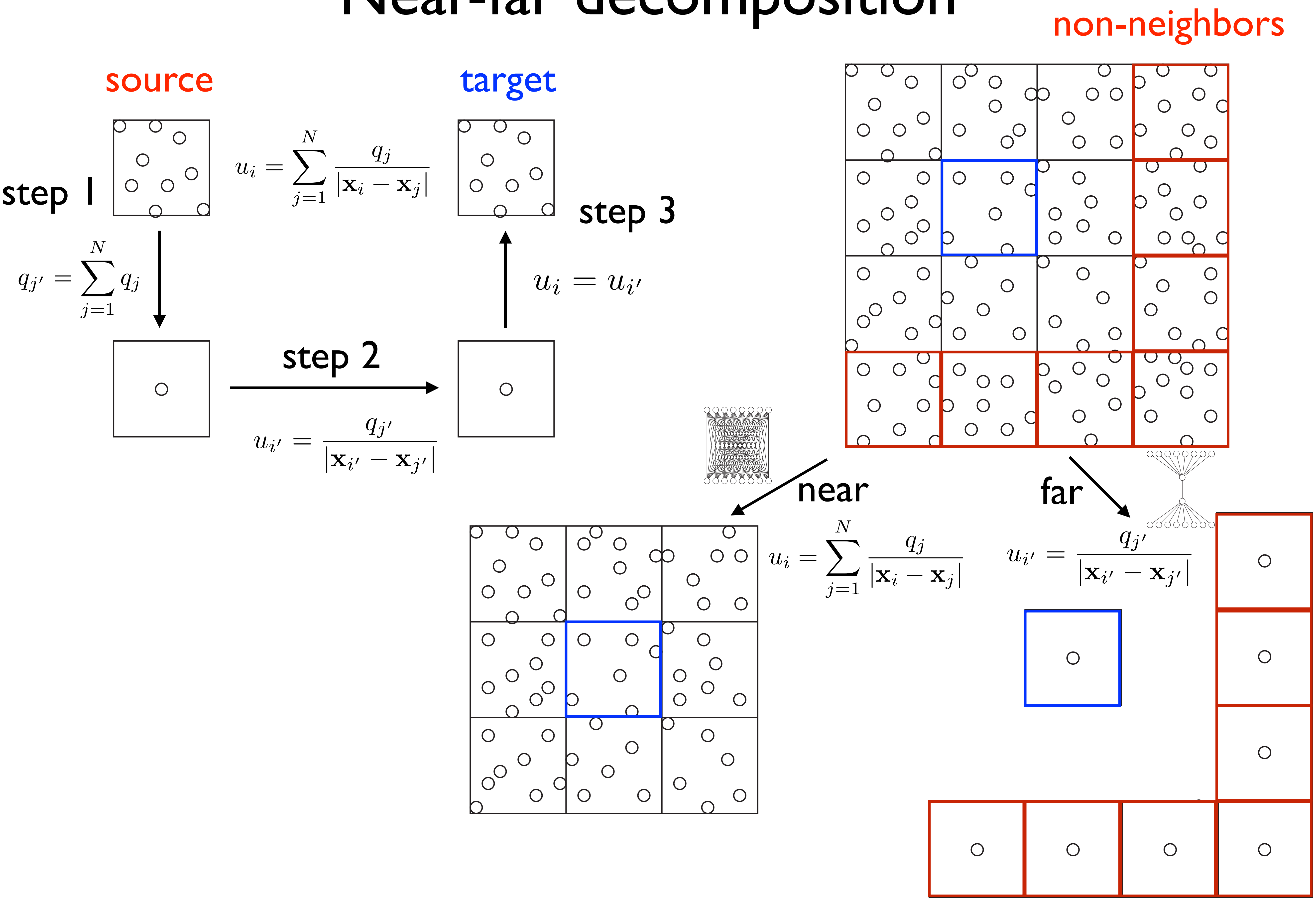
Approximating the interaction



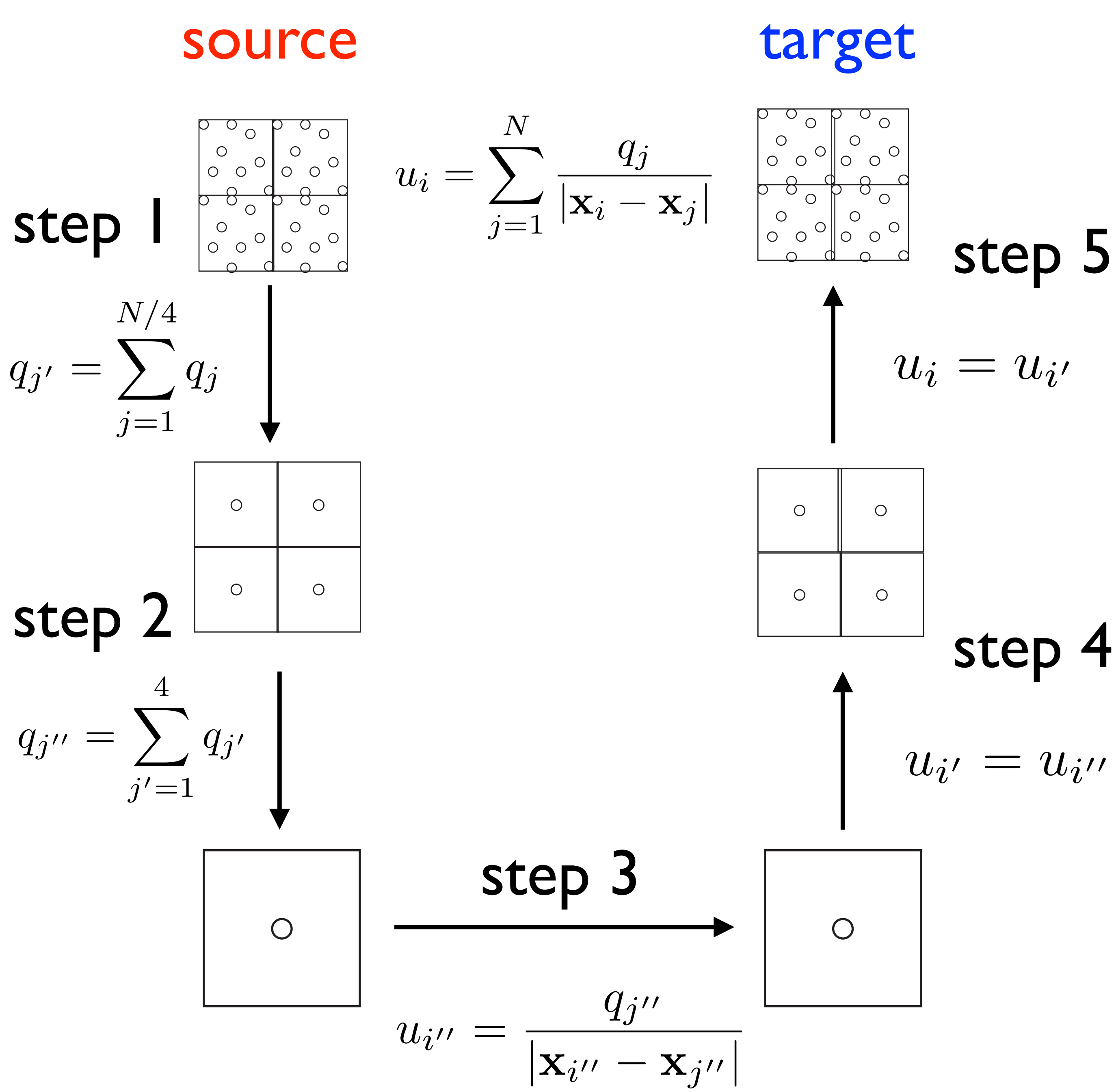


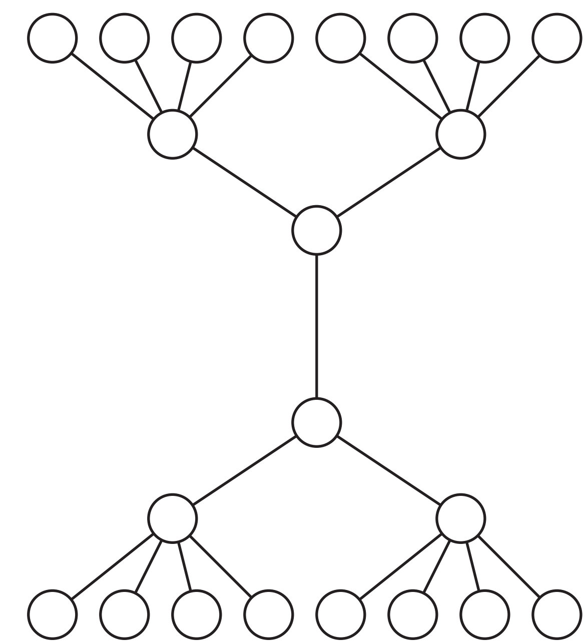
1. Sum all charges
2. Calculate effect of center source on center target
3. Assume all targets in the box have equal potential

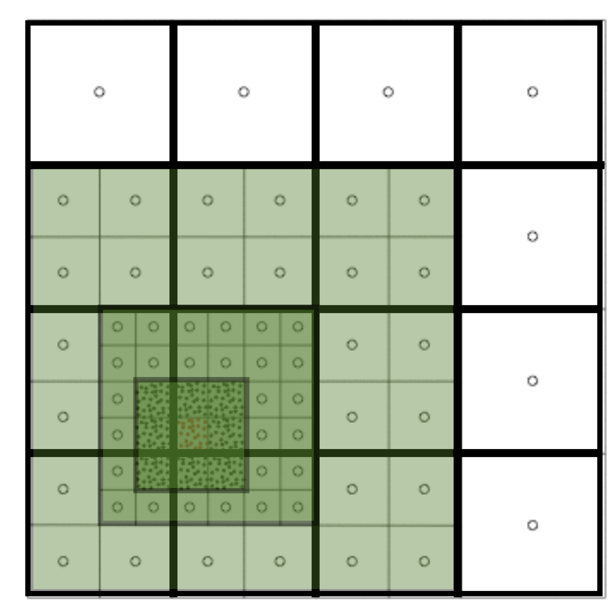
Near-far decomposition



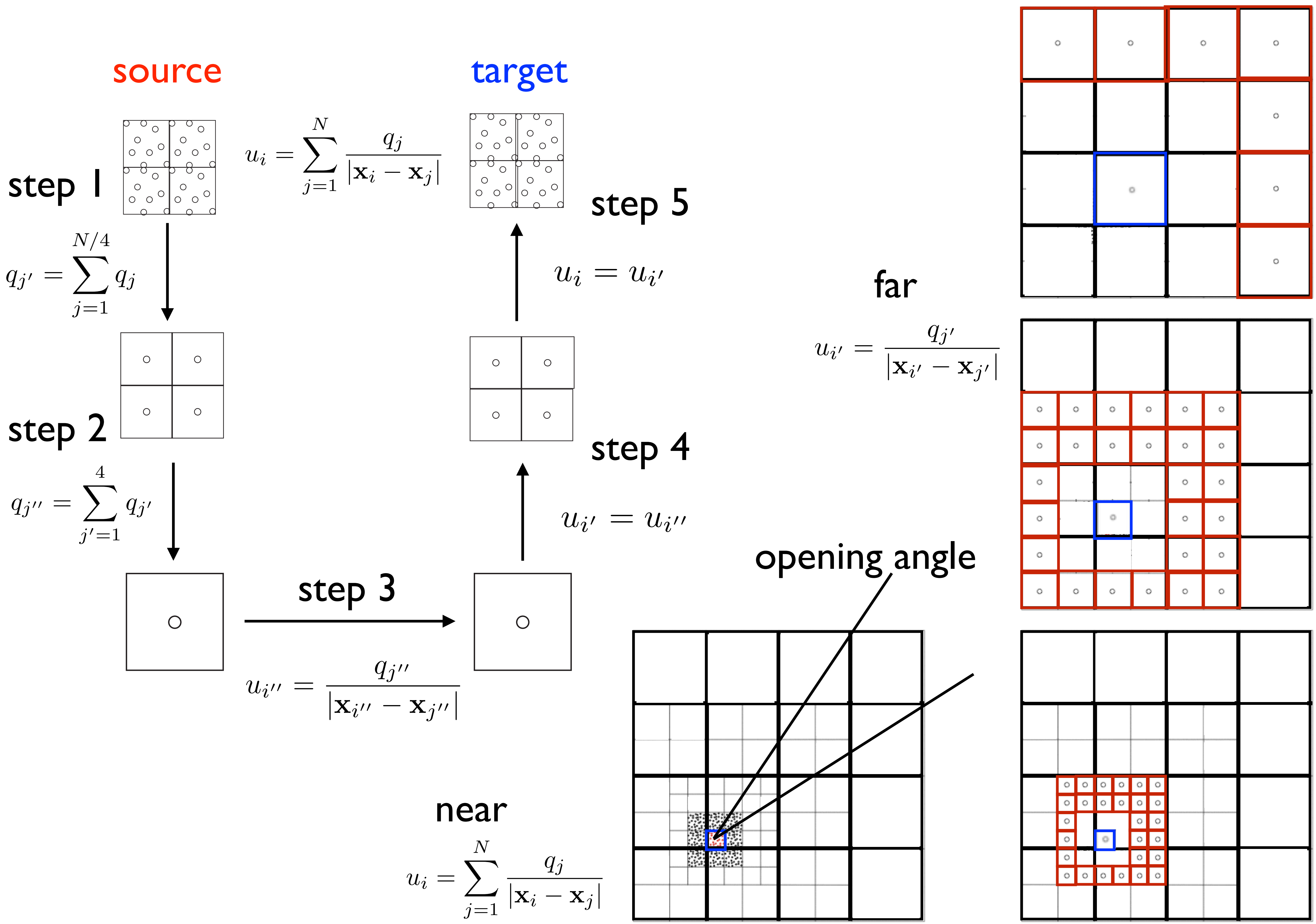
Hierarchical decomposition



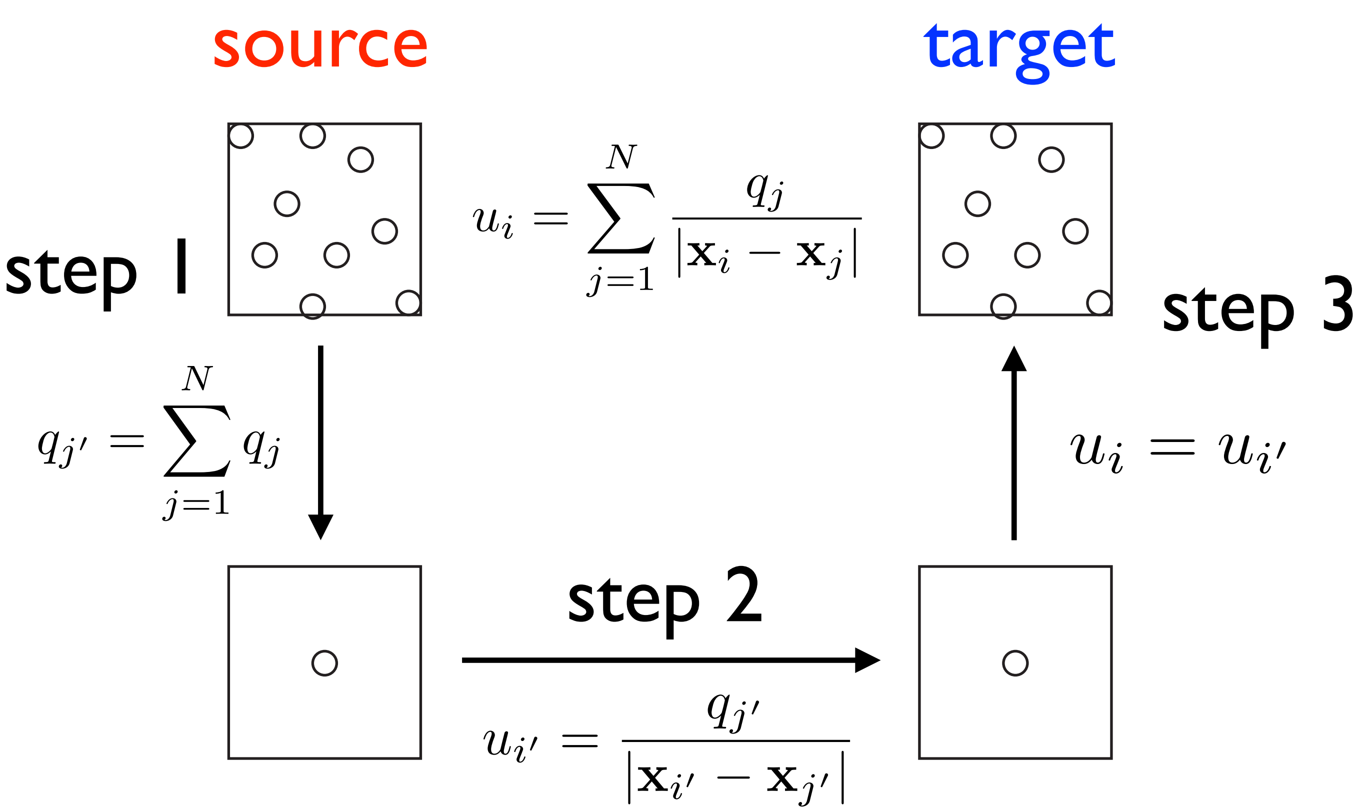


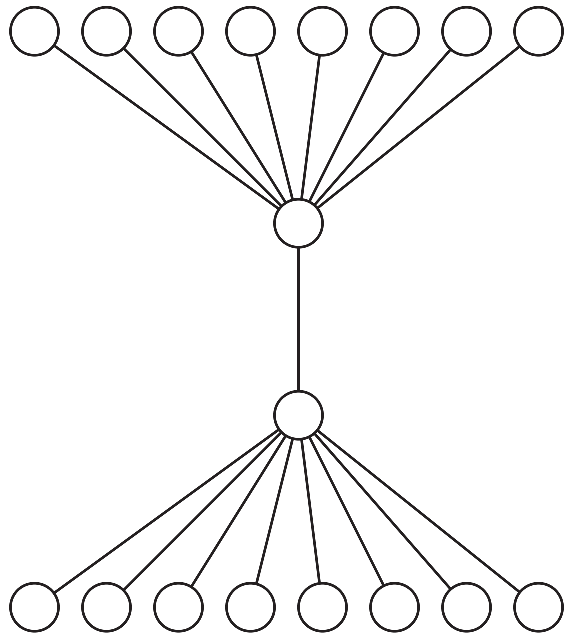


Hierarchical near-far decomposition



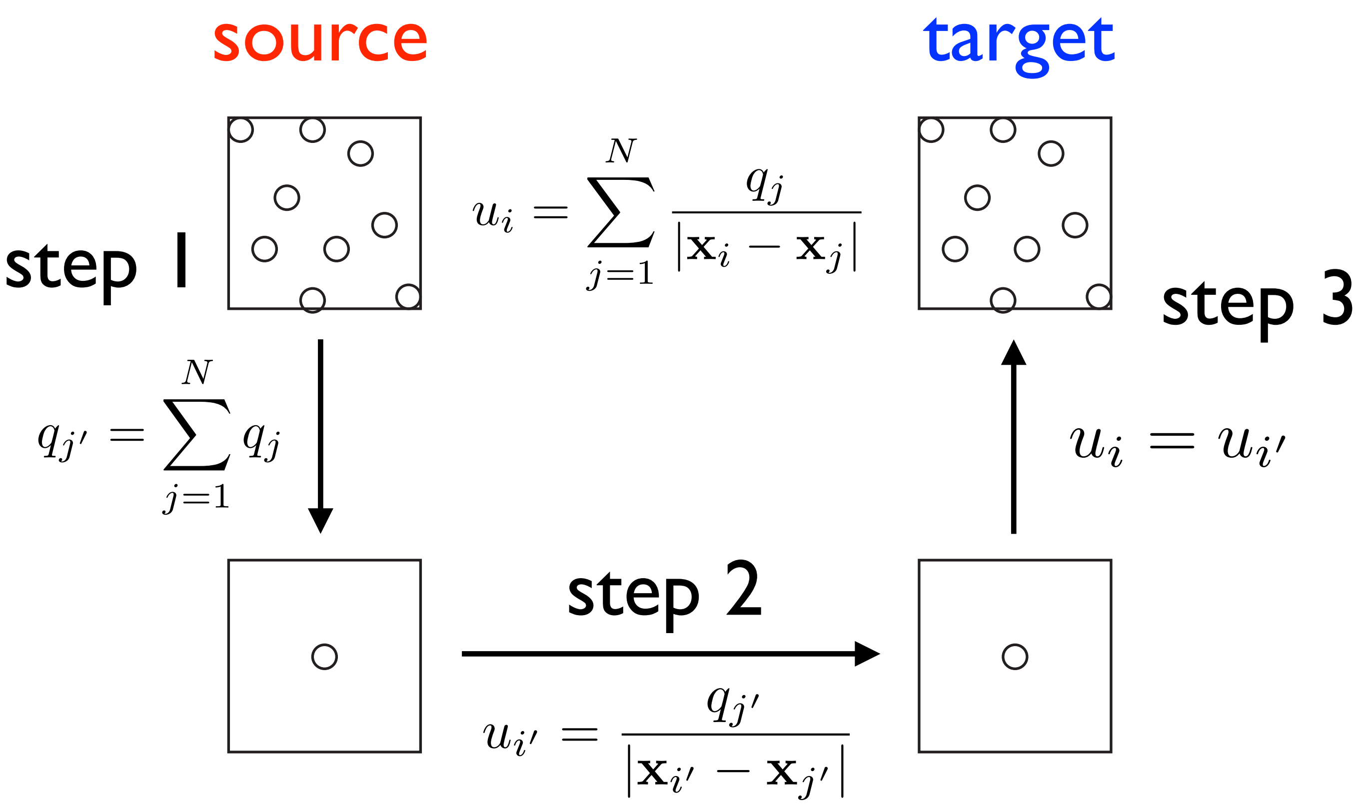
Approximating the interaction

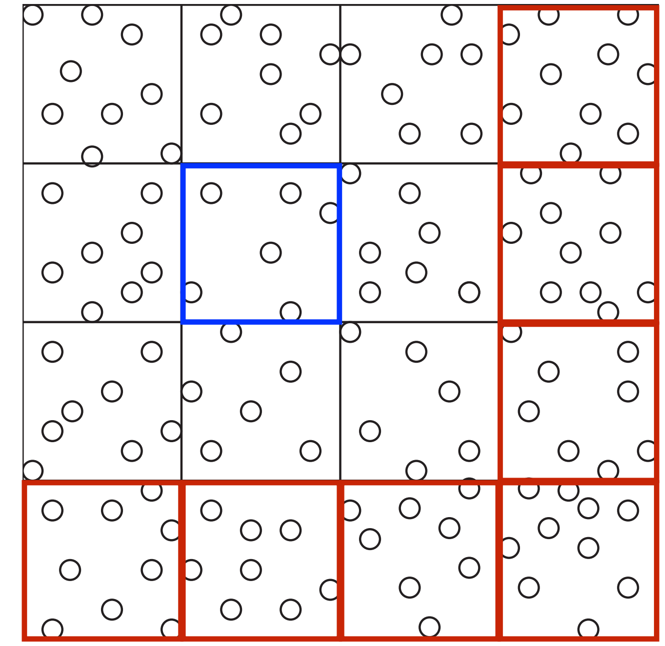




How accurate is the solution?

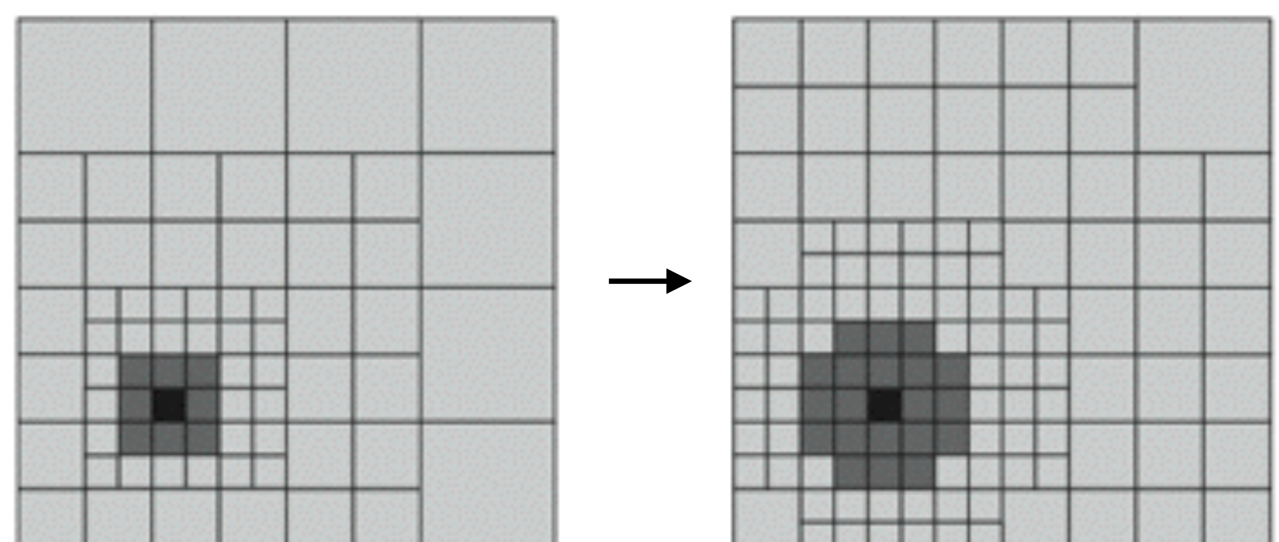
It depends on the well-separatedness





We can separate them more to get an accurate solution

Less approximation



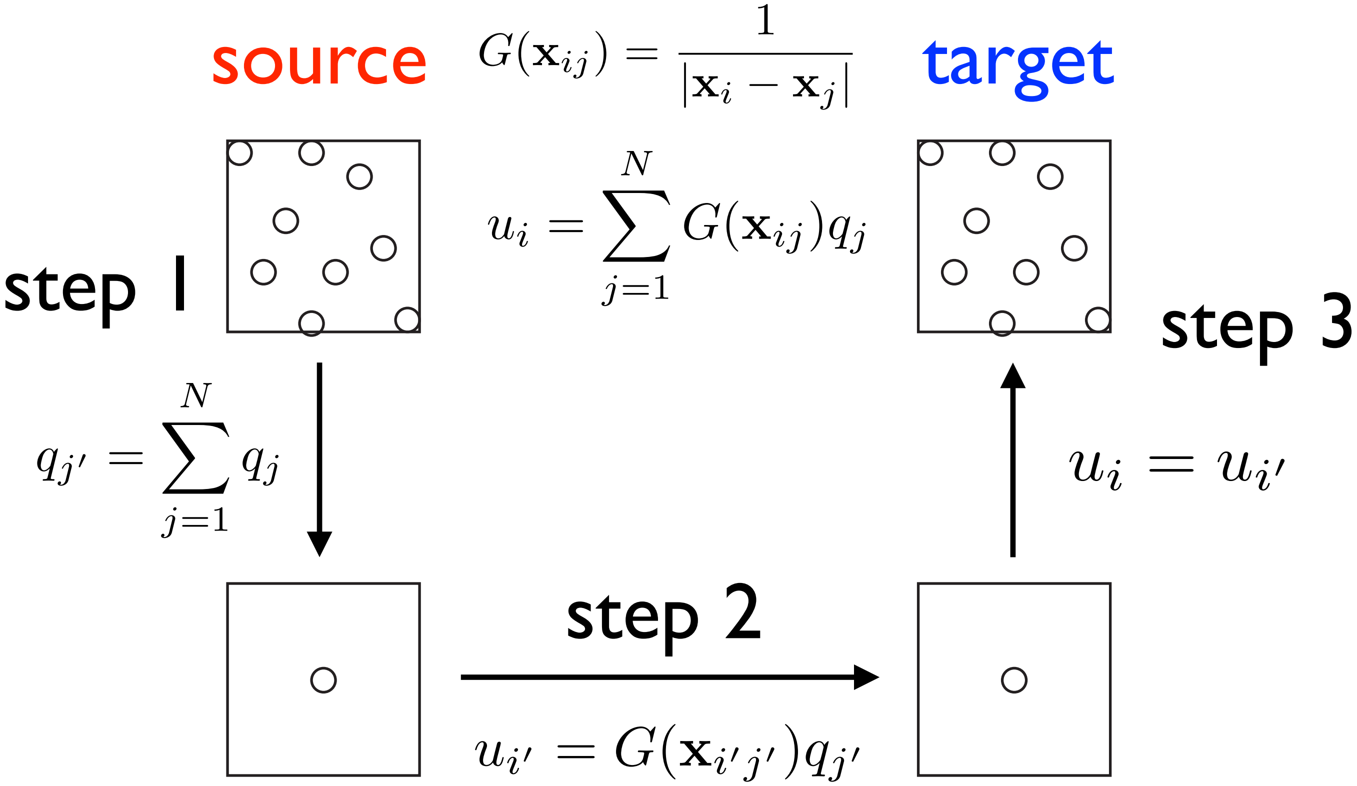
but more boxes to calculate

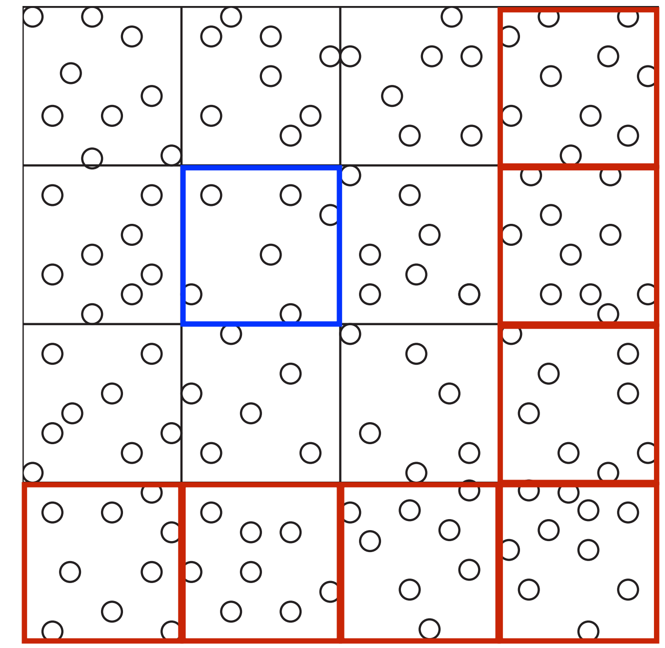
Higher order approximations

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$$

$$G(\mathbf{x}_{ij}) = \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}$$

$$u_i = \sum_{j=1}^{N} G(\mathbf{x}_{ij}) q_j$$





$$\mathbf{x}_{ij} = \mathbf{x}_{ii'} + \mathbf{x}_{i'j'} + \mathbf{x}_{j'j}$$

$$\mathbf{x}_{ii'} + \mathbf{x}_{j'j} \ll \mathbf{x}_{i'j'}$$

$$G(\mathbf{x}_{ij}) = \sum_{\mathbf{n}=0}^{\infty} \frac{1}{\mathbf{n}!} (\mathbf{x}_{ii'} + \mathbf{x}_{j'j})^{\mathbf{n}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'})$$

Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

Binomial theorem
$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^{n-k} y^k$

$$G(\mathbf{x}_{ij}) = \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} (\mathbf{x}_{ii'} + \mathbf{x}_{j'j})^{\mathbf{n}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'})$$

$$\rightarrow = \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'})$$

Cancel $\mathbf{n}!$ $\rightarrow = \sum_{\mathbf{n}=0}^{p} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'})$

Swap loop order between $\mathbf{n}$ and $\mathbf{k}$ $\rightarrow = \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=\mathbf{k}}^{p} \frac{1}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'})$

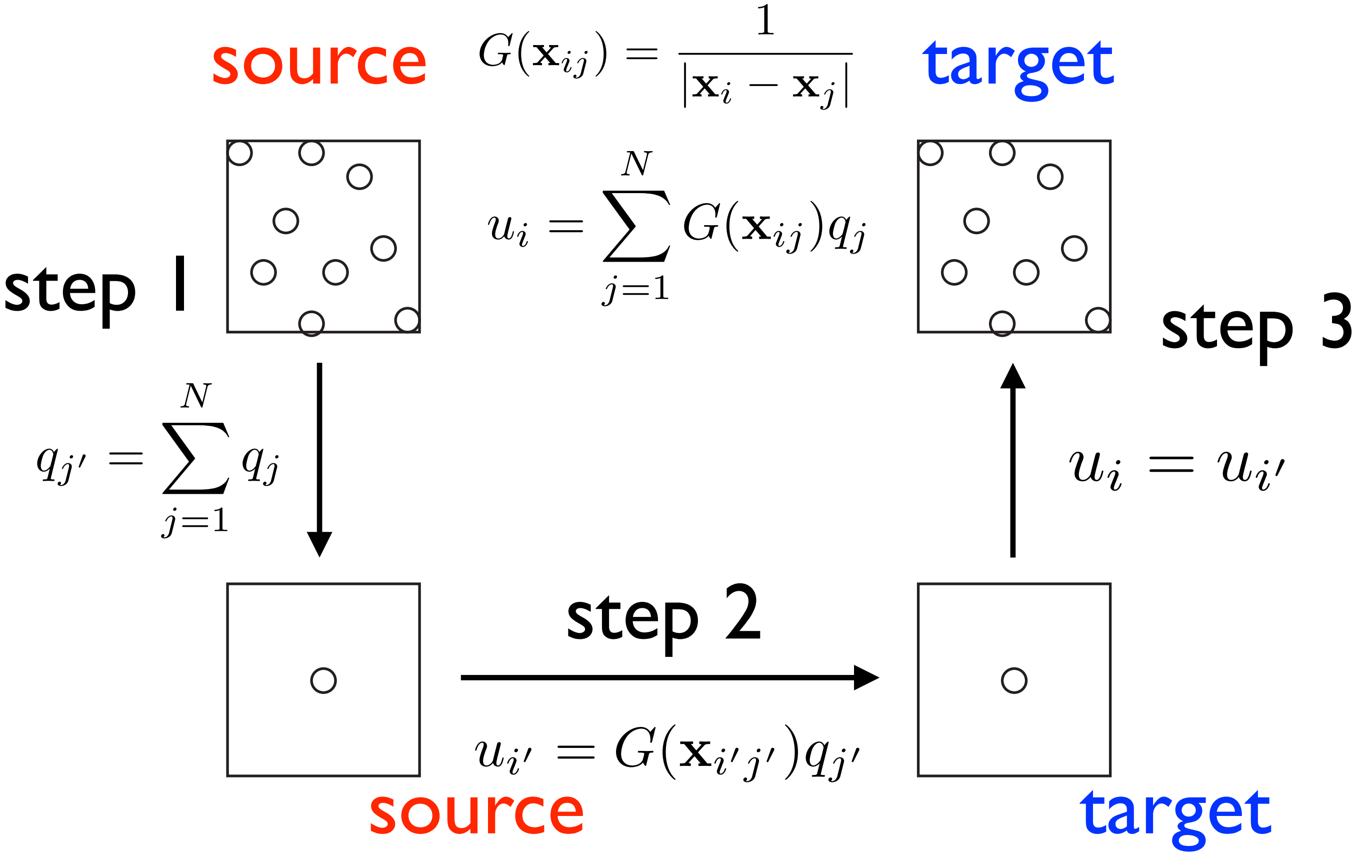
Redefine $\mathbf{n}-\mathbf{k}$ to $\mathbf{n}$ $\rightarrow = \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=0}^{p-\mathbf{k}} \frac{1}{\mathbf{n}!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i'j'})$

$$= \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \underbrace{\sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i'j'}) \underbrace{\frac{1}{\mathbf{n}!} \mathbf{x}_{j'j}^{\mathbf{n}}}_{\mathbf{M}}}_{\mathbf{L}}$$

Higher order approximations

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$$

$$G(\mathbf{x}_{ij}) = \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}$$

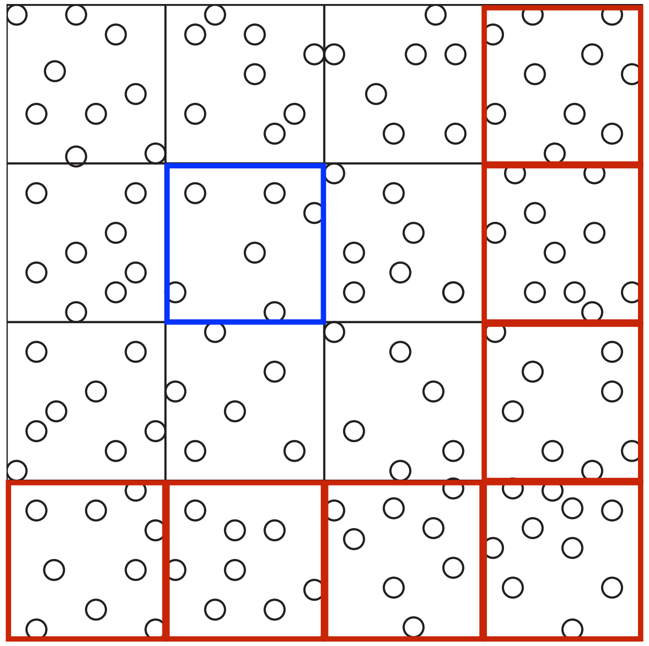


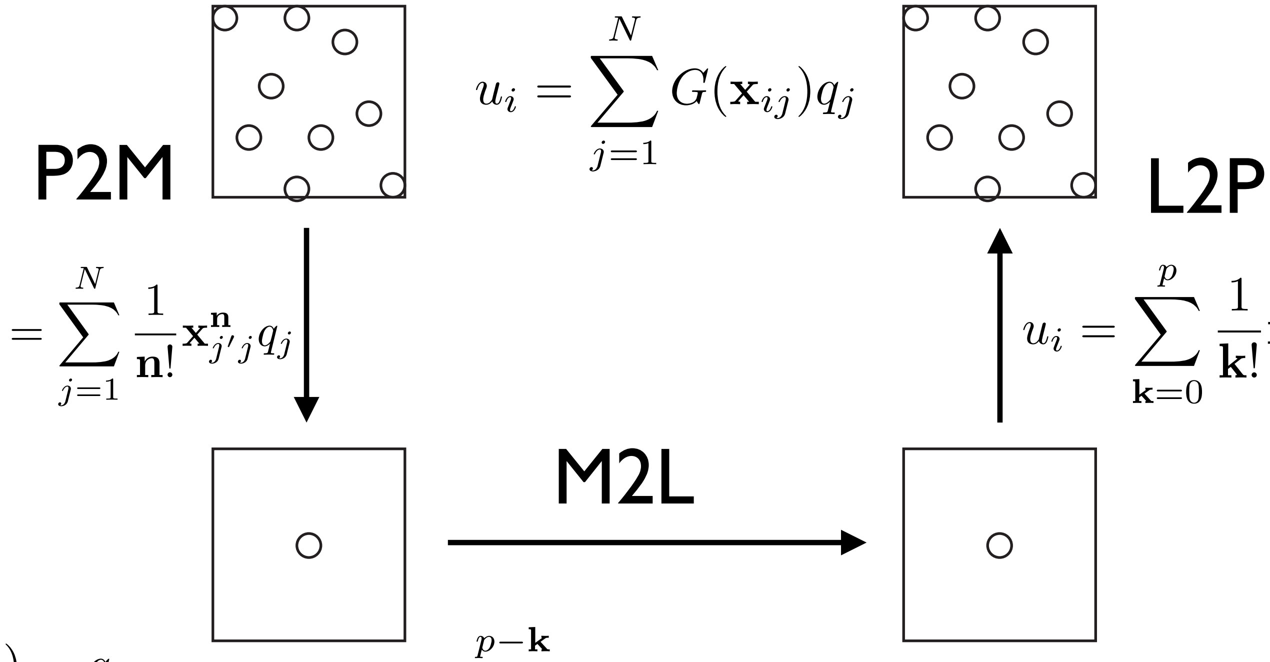
$$u_i = \sum_{j=1}^{N} G(\mathbf{x}_{ij}) q_j$$

$$q_{j'} = \sum_{j=1}^{N} q_j$$

$$u_{i'} = G(\mathbf{x}_{i'j'}) q_{j'}$$

$$u_i = u_{i'}$$





$$u_i = \sum_{j=1}^{N} G(\mathbf{x}_{ij}) q_j$$

$$\mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j'}) = \sum_{j=1}^{N} \frac{1}{\mathbf{n}!} \mathbf{x}_{j'j}^{\mathbf{n}} q_j$$

$$u_i = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i'})$$

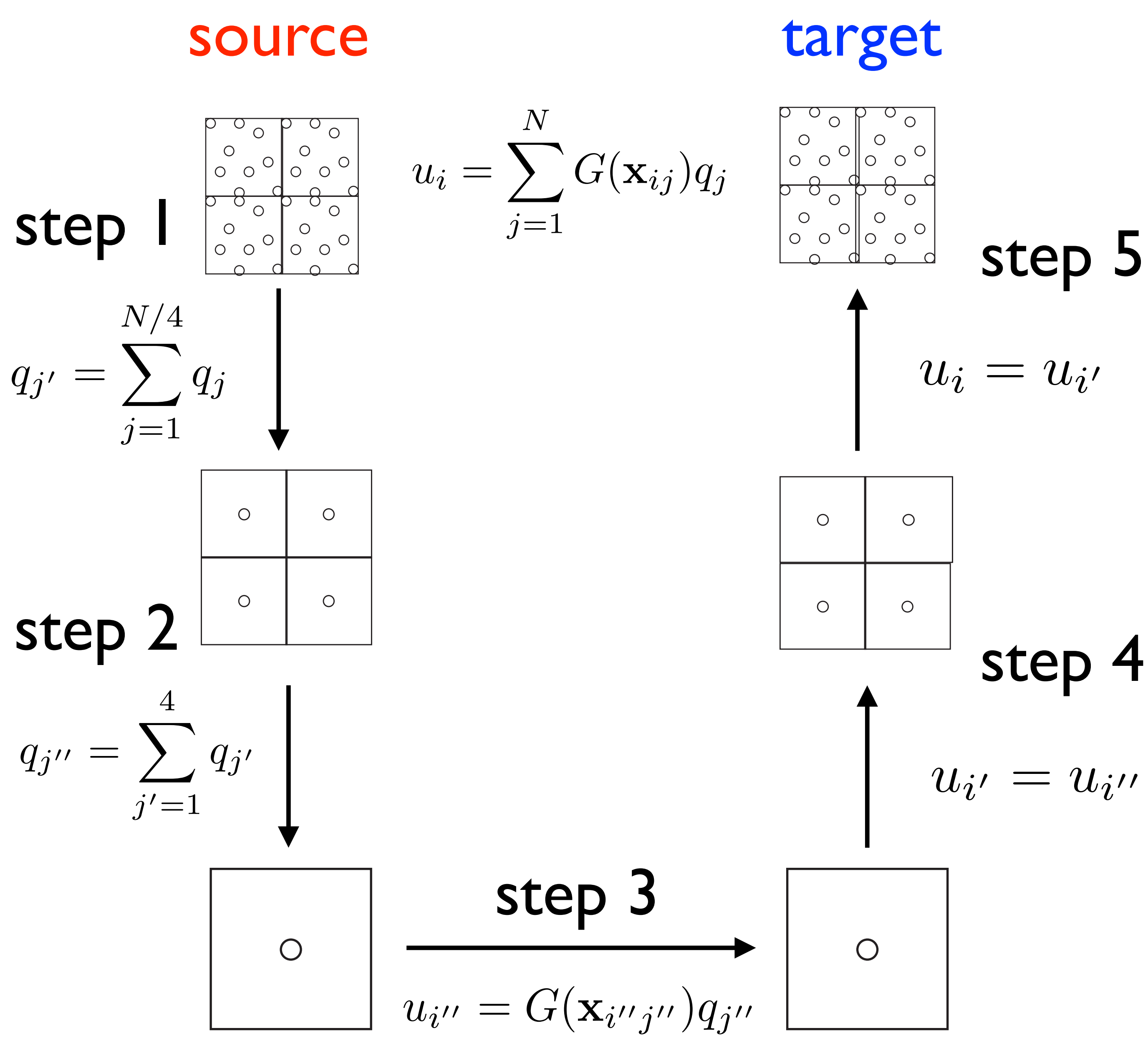
$$\mathbf{M}^0(\mathbf{x}_{j'}) = q_{j'}$$

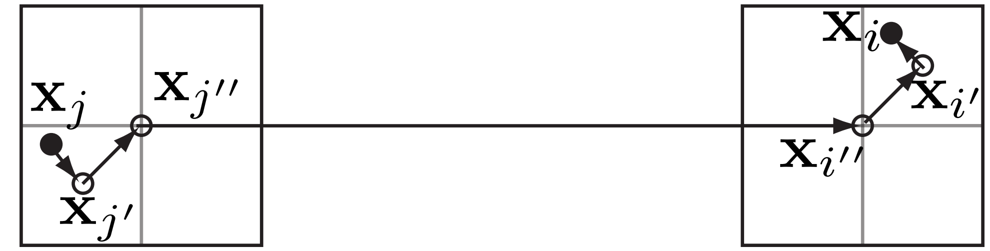
$$\mathbf{L}^0(\mathbf{x}_{i'}) = u_{i'}$$

$$\mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i'}) = \sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i'j'}) \mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j'})$$

$$\begin{aligned}
G(\mathbf{x}_{ij}) &= \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} (\mathbf{x}_{ii'} + \mathbf{x}_{j'j})^{\mathbf{n}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'}) \\
&= \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'}) \\
&= \sum_{\mathbf{n}=0}^{p} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'}) \\
&= \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=\mathbf{k}}^{p} \frac{1}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}-\mathbf{k}} \nabla^{(\mathbf{n})} G(\mathbf{x}_{i'j'}) \\
&= \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=0}^{p-\mathbf{k}} \frac{1}{\mathbf{n}!\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{n}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i'j'}) \\
&= \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \underbrace{\sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i'j'}) \underbrace{\frac{1}{\mathbf{n}!} \mathbf{x}_{j'j}^{\mathbf{n}}}_{\mathbf{M}}}_{\mathbf{L}}
\end{aligned}$$

Multi-level case





$$G(\mathbf{x}_{ij}) = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii''}^{\mathbf{k}} \underbrace{\sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i''j''}) \underbrace{\frac{1}{\mathbf{n}!} \mathbf{x}_{j''j}^{\mathbf{n}}}_{\mathbf{M}}}_{\mathbf{L}}$$

$$\mathbf{x}_{j''j} = \mathbf{x}_{j''j'} + \mathbf{x}_{j'j}$$

$$\mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j''}) = \frac{1}{\mathbf{n}!} (\mathbf{x}_{j''j'} + \mathbf{x}_{j'j})^{\mathbf{n}} q_j$$

$$= \frac{1}{\mathbf{n}!} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{k}} q_j$$

$$= \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{M}^{\mathbf{k}}(\mathbf{x}_{j'})$$

$$\mathbf{x}_{ii''} = \mathbf{x}_{ii'} + \mathbf{x}_{i'i''}$$

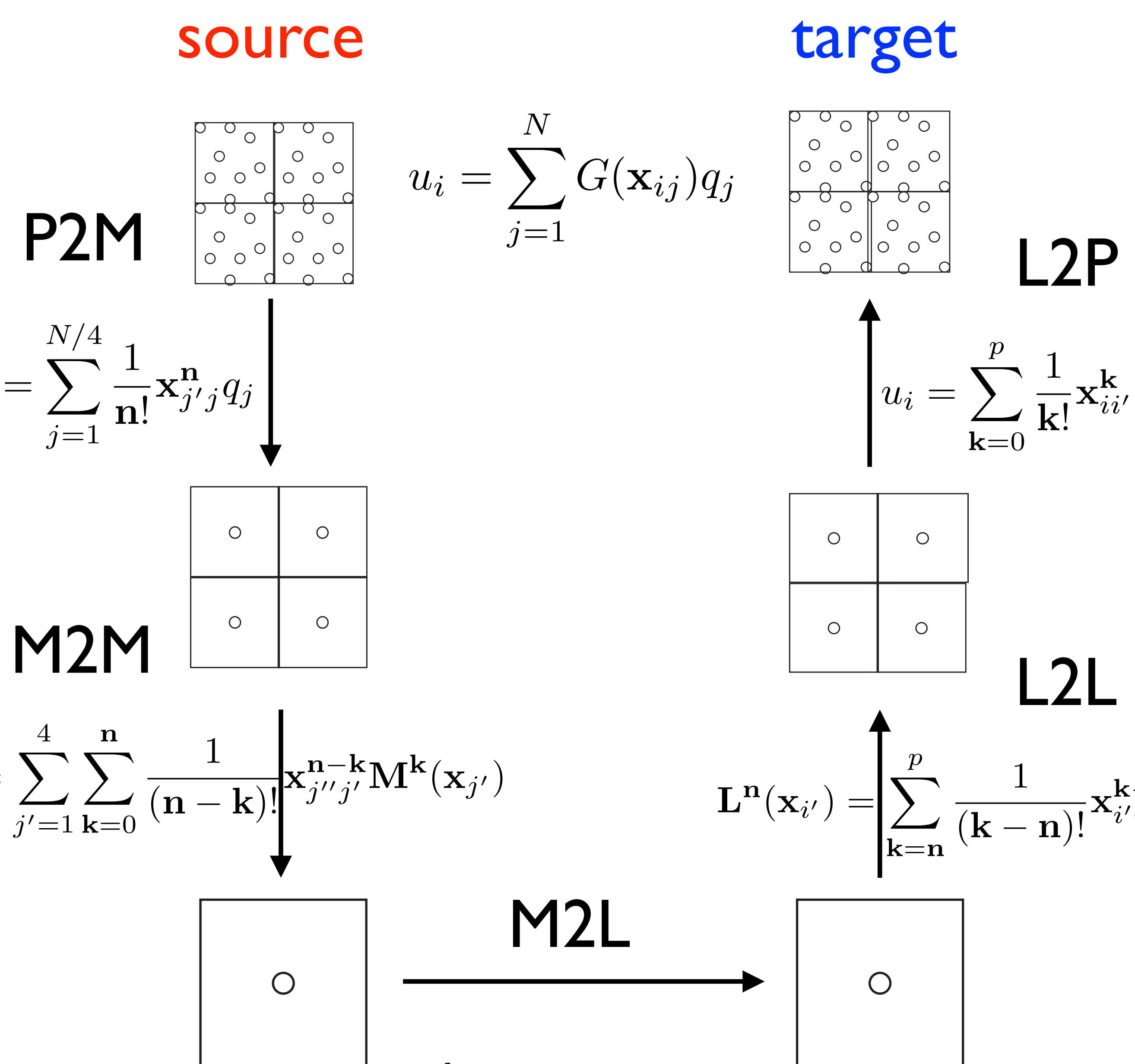
$$u_i = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} (\mathbf{x}_{ii'} + \mathbf{x}_{i'i''})^{\mathbf{k}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''})$$

$$= \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=0}^{\mathbf{k}} \frac{1}{(\mathbf{k}-\mathbf{n})!\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''})$$

$$= \sum_{\mathbf{n}=0}^{p} \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''})$$

$$\mathbf{L}^{\mathbf{n}}(\mathbf{x}_{i'}) = \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''}) \longrightarrow = \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{L}^{\mathbf{n}}(\mathbf{x}_{i'})$$

Multi-level case



$$u_i = \sum_{j=1}^{N} G(\mathbf{x}_{ij}) q_j$$

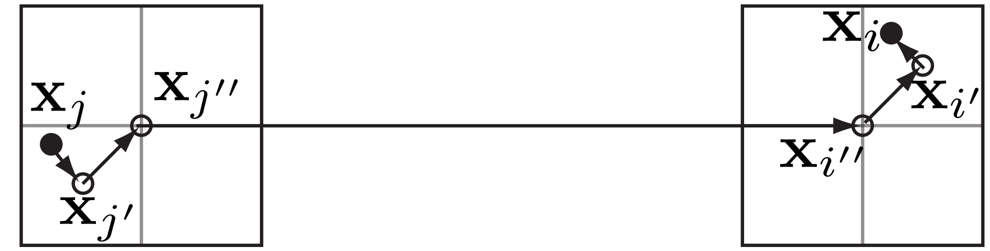
$$\mathbf{M^n}(\mathbf{x}_{j'}) = \sum_{j=1}^{N/4} \frac{1}{\mathbf{n}!} \mathbf{x}_{j'j}^{\mathbf{n}} q_j$$

$$\mathbf{M^n}(\mathbf{x}_{j''}) = \sum_{j'=1}^{4} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{M^k}(\mathbf{x}_{j'})$$

$$\mathbf{L^k}(\mathbf{x}_{i''}) = \sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i''j''}) \mathbf{M^n}(\mathbf{x}_{j''})$$

$$\mathbf{L^n}(\mathbf{x}_{i'}) = \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L^k}(\mathbf{x}_{i''})$$

$$u_i = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{L^k}(\mathbf{x}_{i'})$$



$$G(\mathbf{x}_{ij}) = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii''}^{\mathbf{k}} \underbrace{\sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i''j''}) \underbrace{\frac{1}{\mathbf{n}!} \mathbf{x}_{j''j}^{\mathbf{n}}}_{\mathbf{M}}}_{\mathbf{L}}$$

$$\mathbf{x}_{j''j} = \mathbf{x}_{j''j'} + \mathbf{x}_{j'j}$$

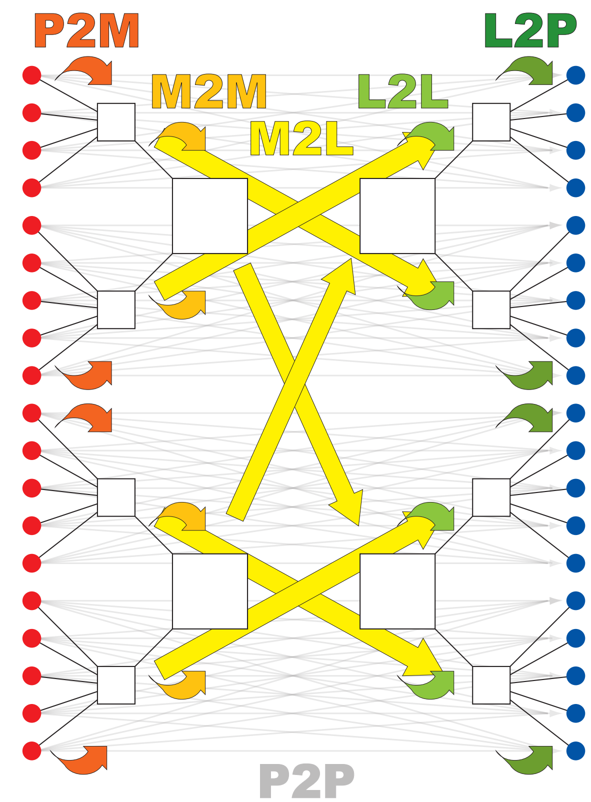
$$\begin{aligned} \mathbf{M^n}(\mathbf{x}_{j''}) &= \frac{1}{\mathbf{n}!} (\mathbf{x}_{j''j'} + \mathbf{x}_{j'j})^{\mathbf{n}} q_j \\ &= \frac{1}{\mathbf{n}!} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!\mathbf{k}!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{x}_{j'j}^{\mathbf{k}} q_j \\ &= \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{M^k}(\mathbf{x}_{j'}) \end{aligned}$$

$$\mathbf{x}_{ii''} = \mathbf{x}_{ii'} + \mathbf{x}_{i'i''}$$

$$\begin{aligned} u_i &= \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} (\mathbf{x}_{ii'} + \mathbf{x}_{i'i''})^{\mathbf{k}} \mathbf{L^k}(\mathbf{x}_{i''}) \\ &= \sum_{\mathbf{k}=0}^{p} \sum_{\mathbf{n}=0}^{\mathbf{k}} \frac{1}{(\mathbf{k}-\mathbf{n})!\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L^k}(\mathbf{x}_{i''}) \\ &= \sum_{\mathbf{n}=0}^{p} \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L^k}(\mathbf{x}_{i''}) \end{aligned}$$

$$\mathbf{L^n}(\mathbf{x}_{i'}) = \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L^k}(\mathbf{x}_{i''}) \longrightarrow = \sum_{\mathbf{n}=0}^{p} \frac{1}{\mathbf{n}!} \mathbf{x}_{ii'}^{\mathbf{n}} \mathbf{L^n}(\mathbf{x}_{i'})$$

Flow of Calculation



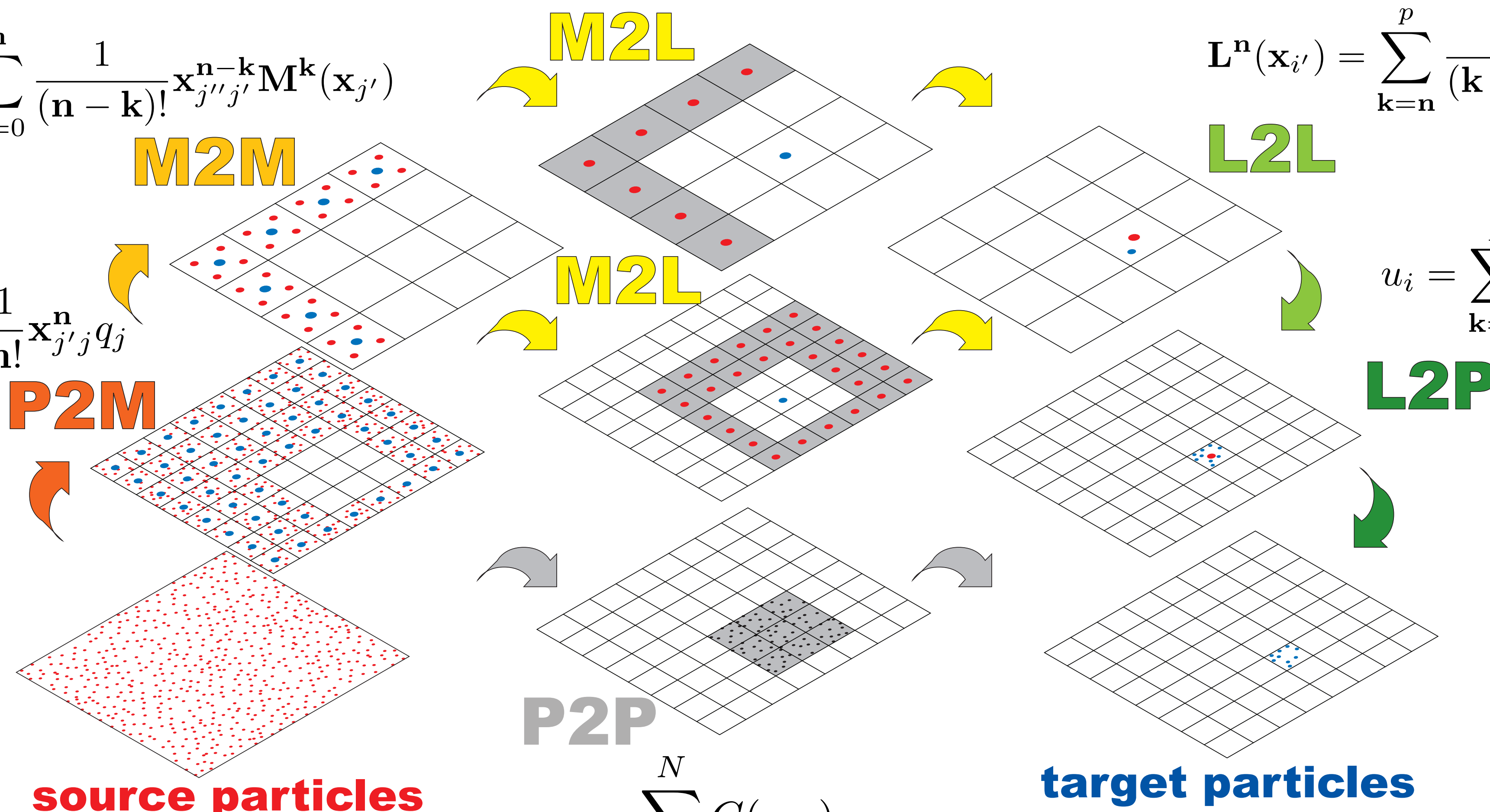
$$\mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''}) = \sum_{\mathbf{n}=0}^{p-\mathbf{k}} \nabla^{(\mathbf{n}+\mathbf{k})} G(\mathbf{x}_{i''j''}) \mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j''})$$

$$\mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j''}) = \sum_{j'=1}^{4} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(\mathbf{n}-\mathbf{k})!} \mathbf{x}_{j''j'}^{\mathbf{n}-\mathbf{k}} \mathbf{M}^{\mathbf{k}}(\mathbf{x}_{j'})$$

$$\mathbf{L}^{\mathbf{n}}(\mathbf{x}_{i'}) = \sum_{\mathbf{k}=\mathbf{n}}^{p} \frac{1}{(\mathbf{k}-\mathbf{n})!} \mathbf{x}_{i'i''}^{\mathbf{k}-\mathbf{n}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i''})$$

$$u_i = \sum_{\mathbf{k}=0}^{p} \frac{1}{\mathbf{k}!} \mathbf{x}_{ii'}^{\mathbf{k}} \mathbf{L}^{\mathbf{k}}(\mathbf{x}_{i'})$$

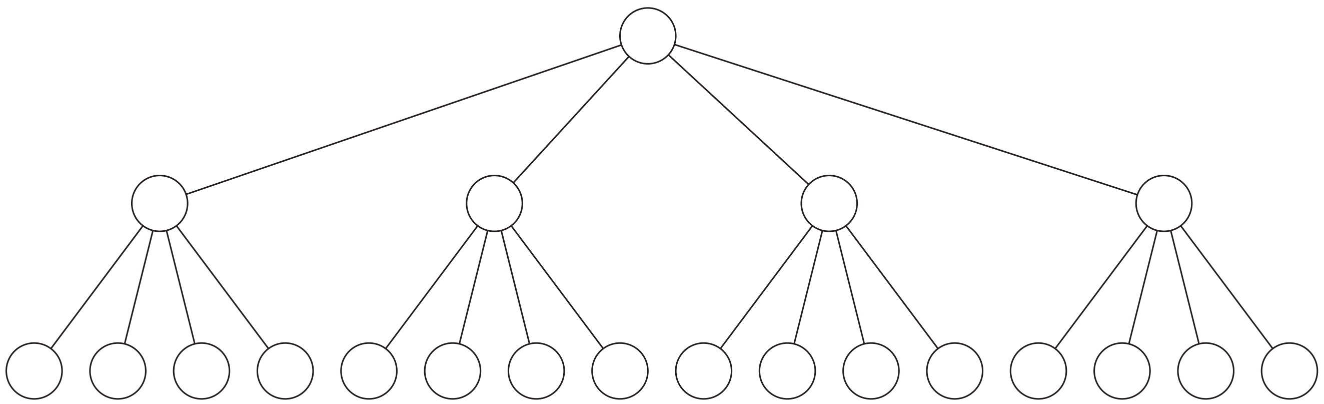
$$\mathbf{M}^{\mathbf{n}}(\mathbf{x}_{j'}) = \sum_{j=1}^{N/4} \frac{1}{\mathbf{n}!} \mathbf{x}_{j'j}^{\mathbf{n}} q_j$$



$$u_i = \sum_{j=1}^{N} G(\mathbf{x}_{ij}) q_j$$

Tree Structure

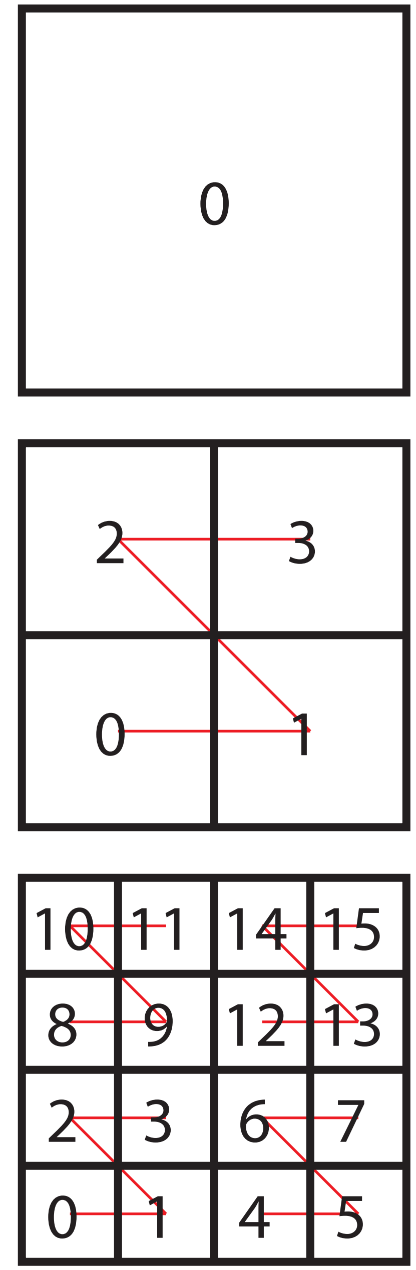
Linked tree



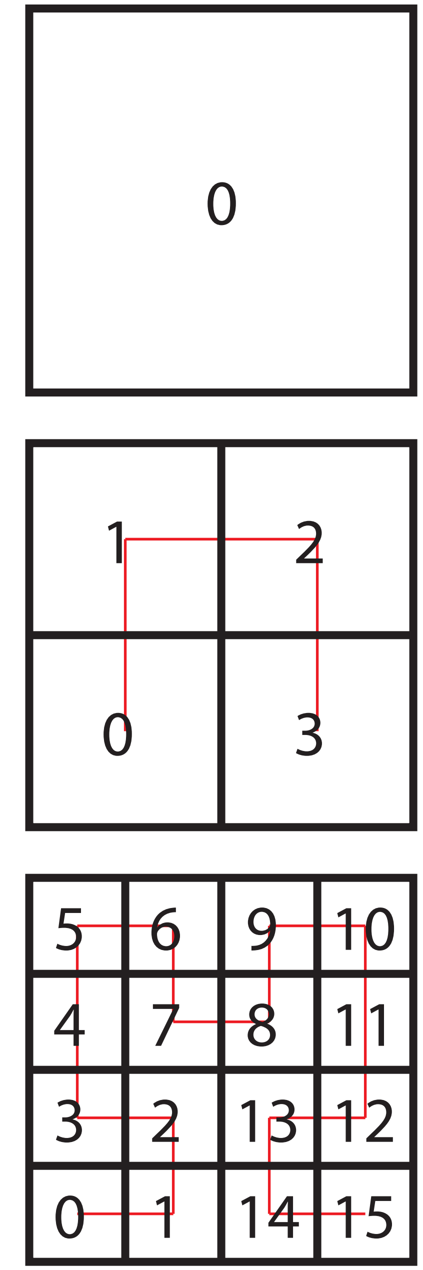
```
struct Cell {
  Cell *parent;
  Cell *child[4];
  double Multipole;
}
```

Indexed tree

Morton

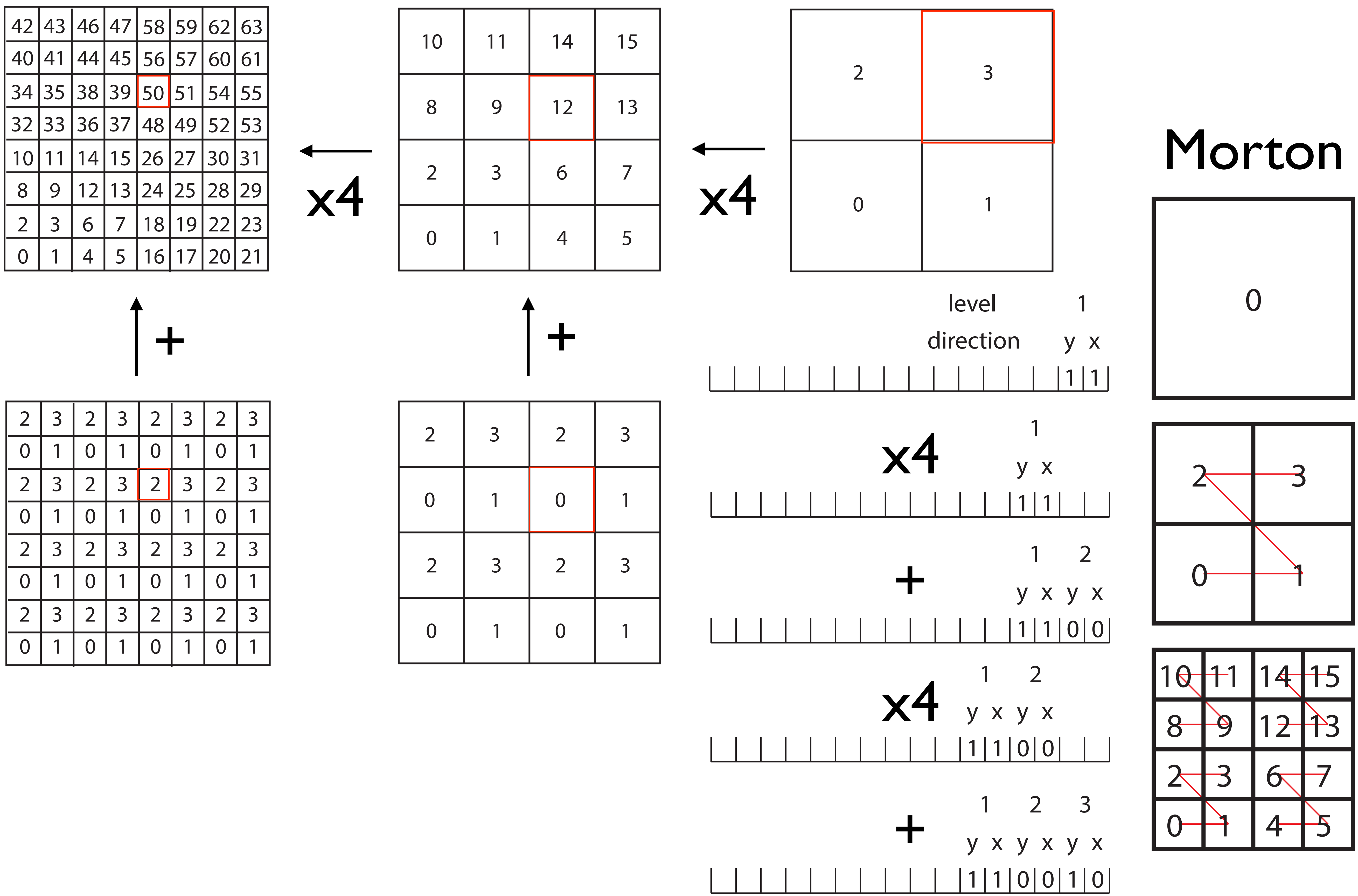


Hilbert

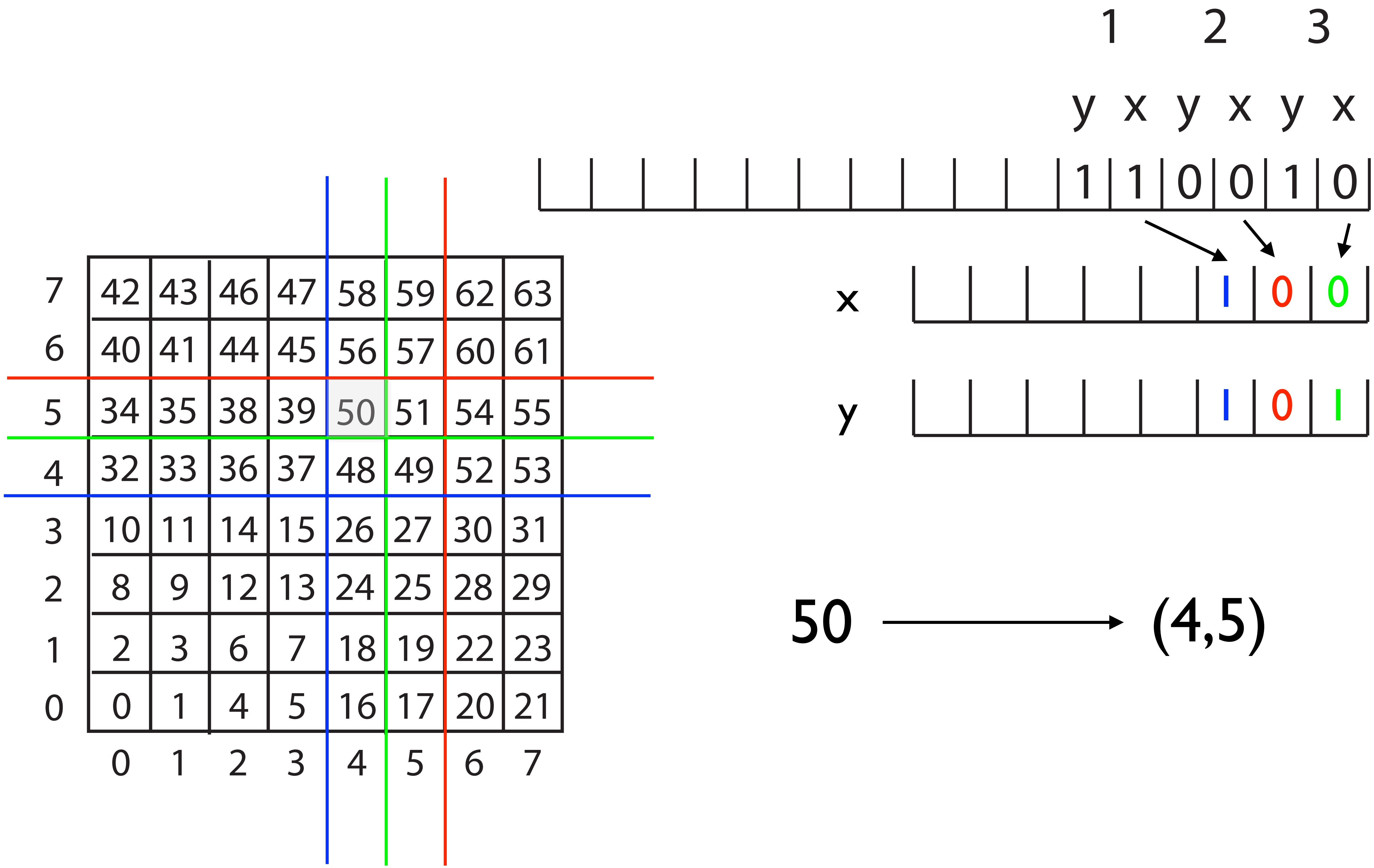



```
int parent = i/4;
int child[c] = i*4+c;
Multipole[i]
```

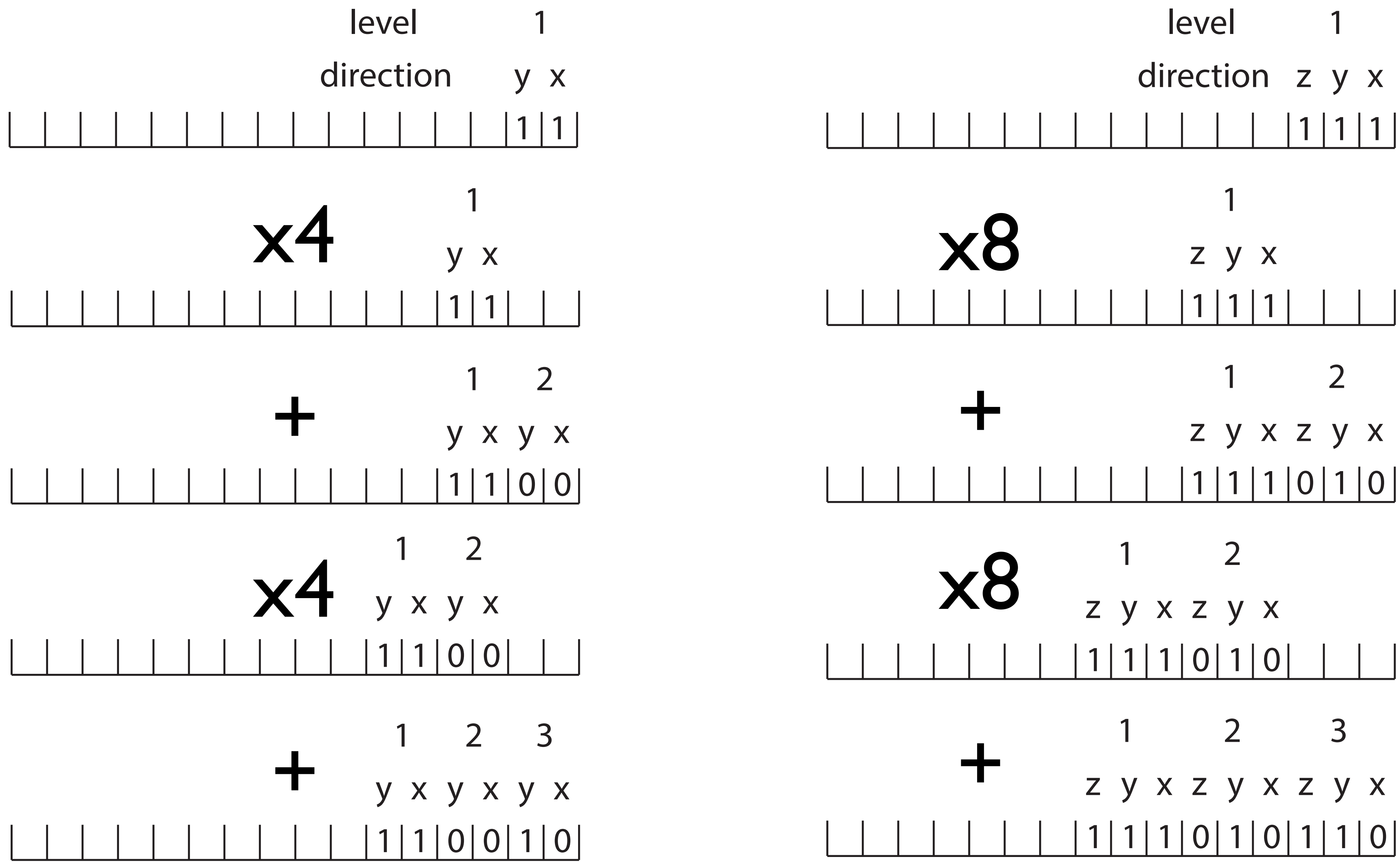
Bit interleaving and space filling curves



Interleaving the bits



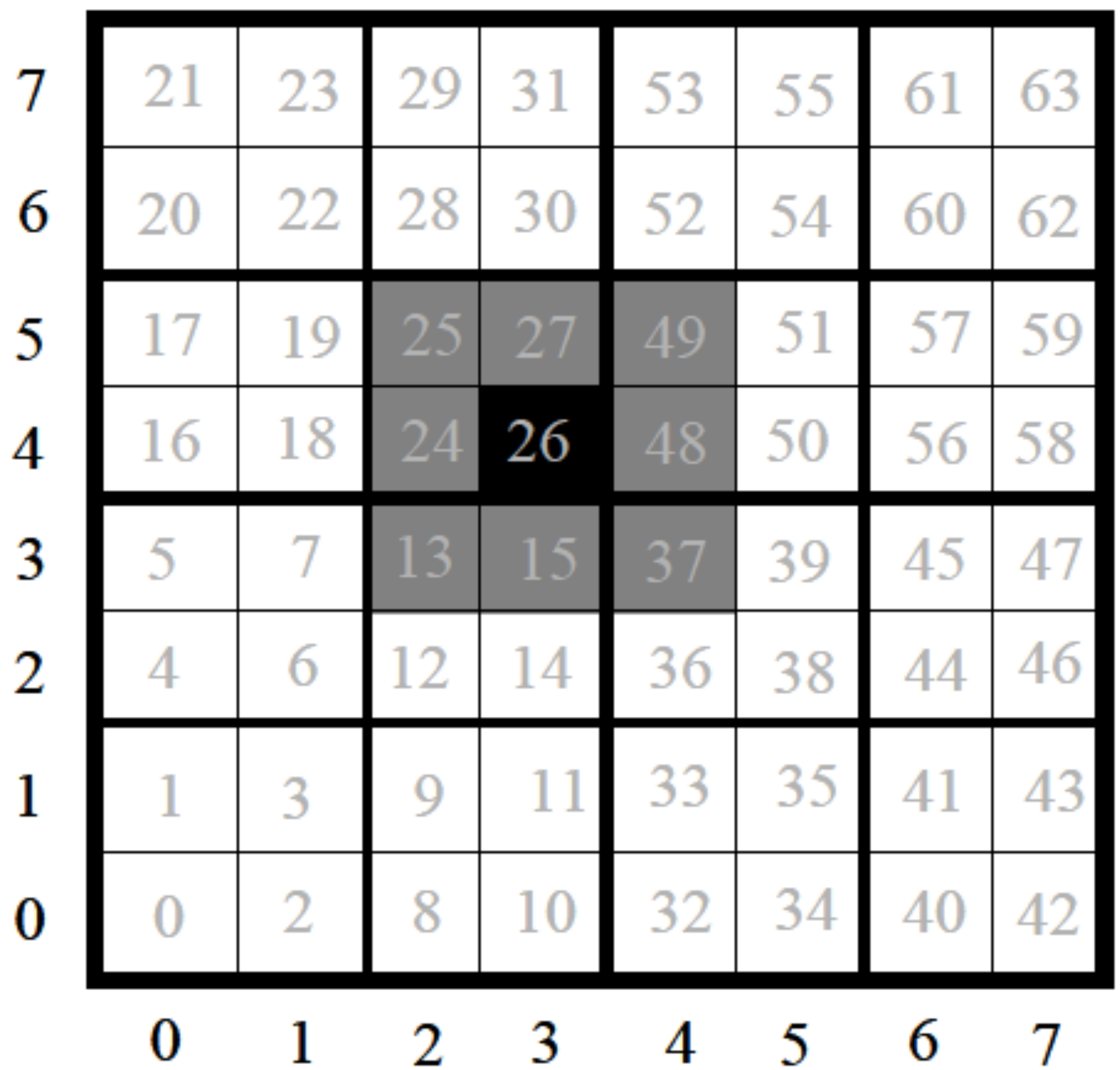
3-D version



(ix,iy,iz) —interleave bits→ i

(ix,iy,iz) ←deinterleave bits— i

Neighbor finding



Find neighbors of 26

Deinterleave 26 → (3,4)

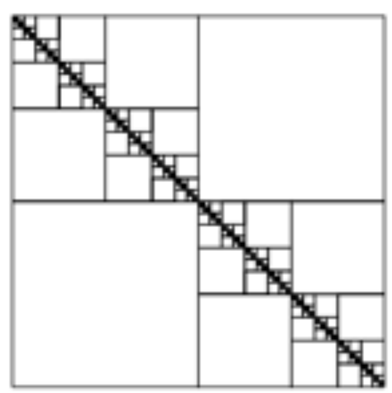
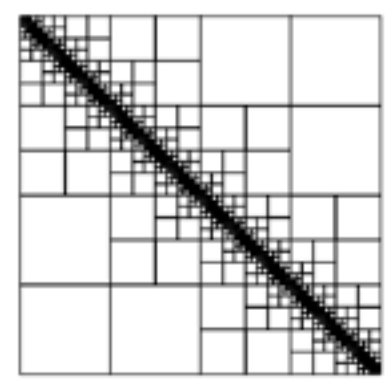
Neighbors are
(2,5),(3,5),(4,5)
(2,4),(3,4),(4,4)
(2,3),(3,3),(4,3)

↓

Interleave
25,27,49
24,26,48
13,15,37

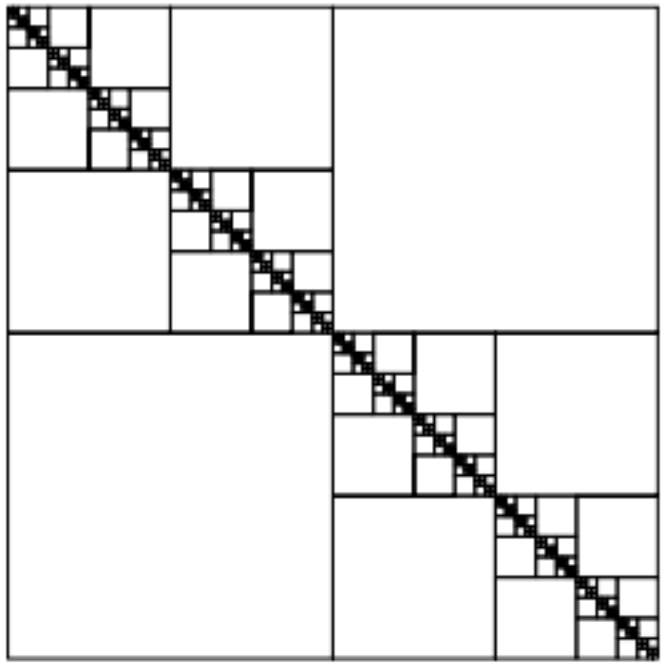
(ix,iy,iz) —interleave bits→ i

(ix,iy,iz) ←deinterleave bits— i

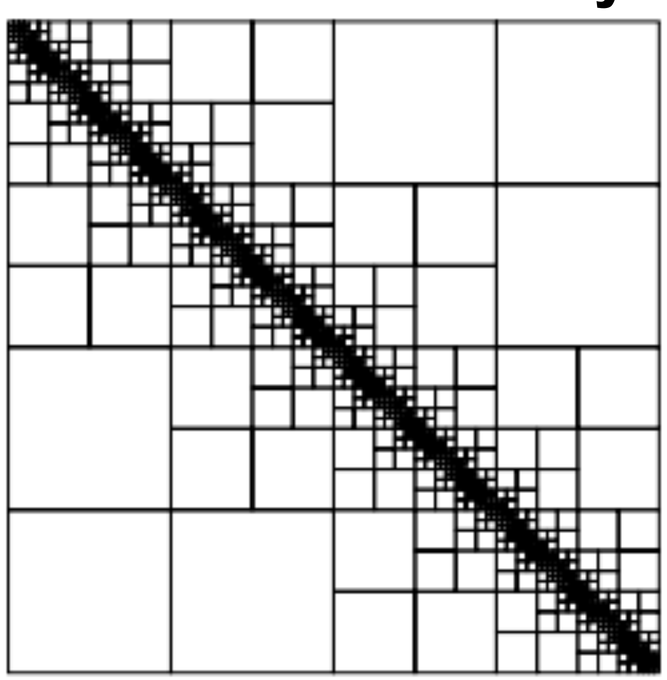
HSS	FMM
Stores matrix (algebraic) Good for multiple r.h.s. Easy to factorize matrix	Matrix-free (geometric) Low memory footprint High arithmetic intensity
Weak admissibility 1-D or 2-D problems  **HSS2D**: Xia, J. **HIF**:Ho.K&Ying.L	Standard admissibility High dimension problems  **IFMM**: Ambikasaran, S & Darve, E.
Requires: Numerically low-rank off-diagonal blocks	Requires: analytical form (doesn't have to be a Green's function) e.g. kernel-independent/black-box FMM

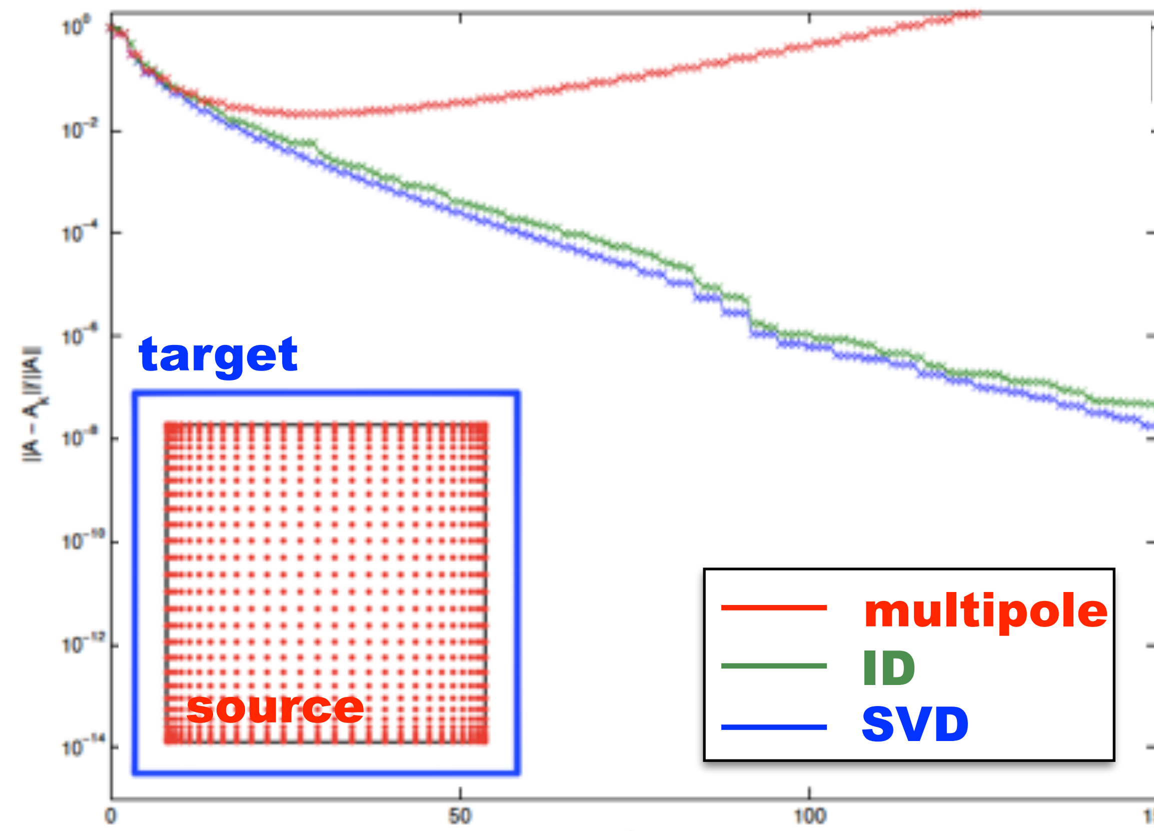
Admissibility condition

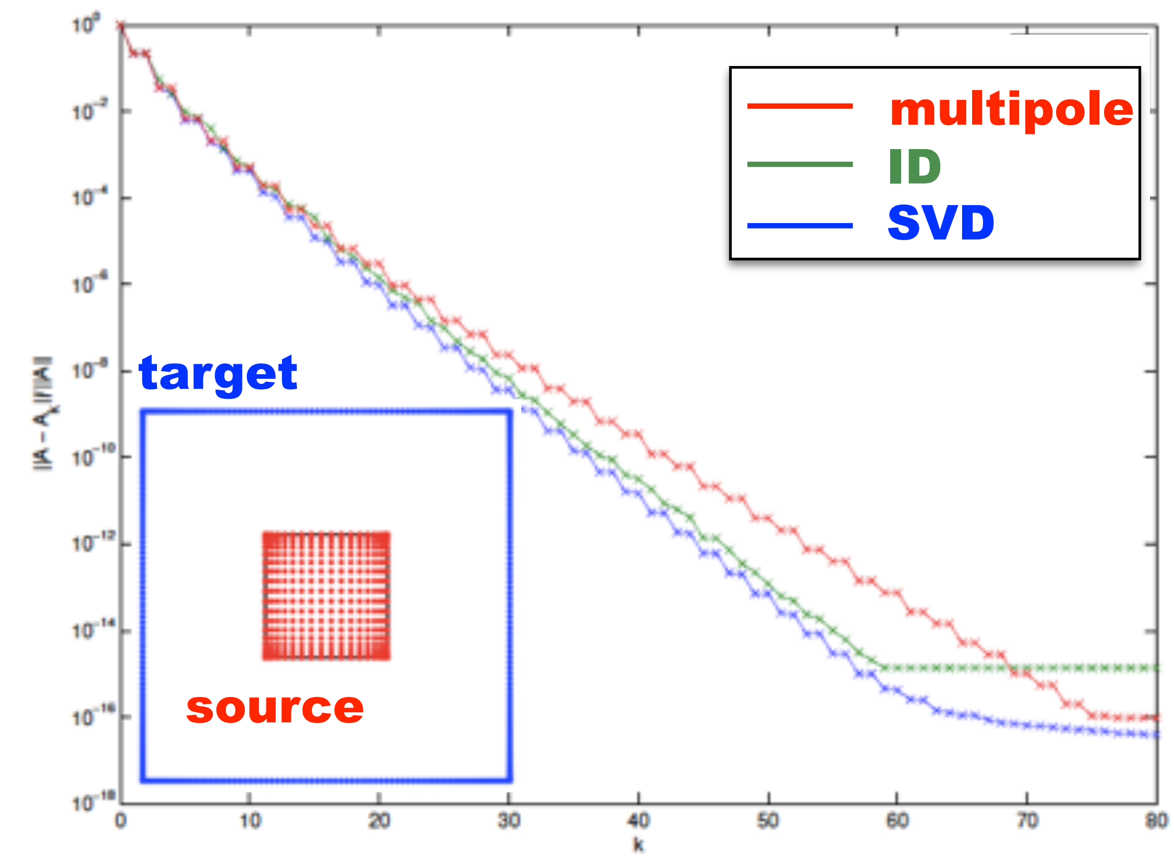
Weak admissibility



Standard admissibility

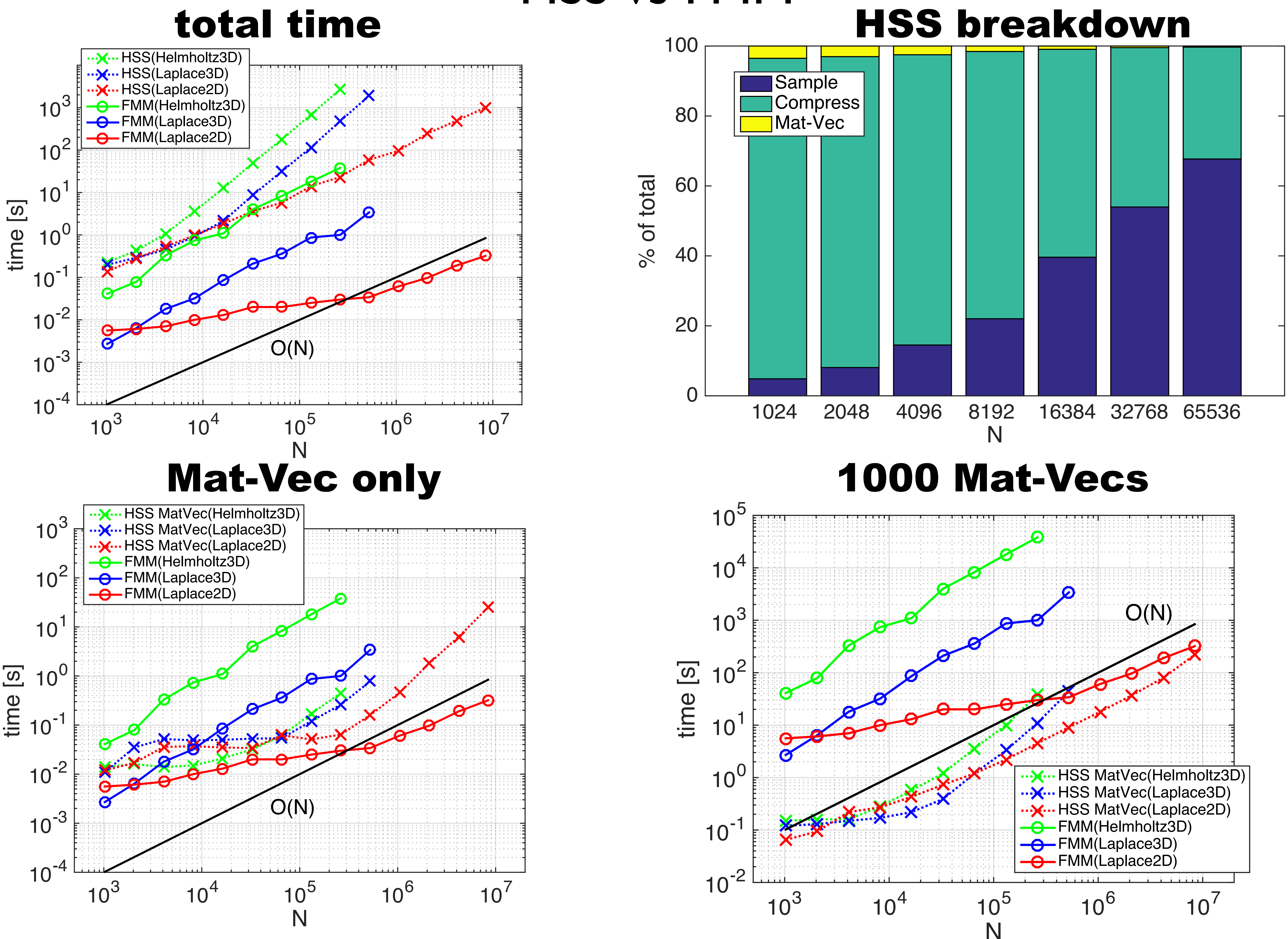






P. G. Martinsson, CBMS/NSF Conference on Fast Direct Solvers, June 2014

HSS vs FMM


total time
HSS(Helmholtz3D)
HSS(Laplace3D)
HSS(Laplace2D)
FMM(Helmholtz3D)
FMM(Laplace3D)
FMM(Laplace2D)
O(N)
time [s]
N
HSS breakdown
Sample
Compress
Mat-Vec
% of total
1024
2048
4096
8192
16384
32768
65536
N
Mat-Vec only
HSS MatVec(Helmholtz3D)
HSS MatVec(Laplace3D)
HSS MatVec(Laplace2D)
FMM(Helmholtz3D)
FMM(Laplace3D)
FMM(Laplace2D)
O(N)
time [s]
N
1000 Mat-Vecs
O(N)
HSS MatVec(Helmholtz3D)
HSS MatVec(Laplace3D)
HSS MatVec(Laplace2D)
FMM(Helmholtz3D)
FMM(Laplace3D)
FMM(Laplace2D)
time [s]
N