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04/21	Class 5	Boundary element methods	Understand the relation between inverse matrices, δ functions and Green's functions, and solve boundary integral equations.
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04/28	Class 7	Smooth particle hydrodynamics (SPH)	Evaluate the conservation and dissipation properties of differential operators formed from radial basis functions.
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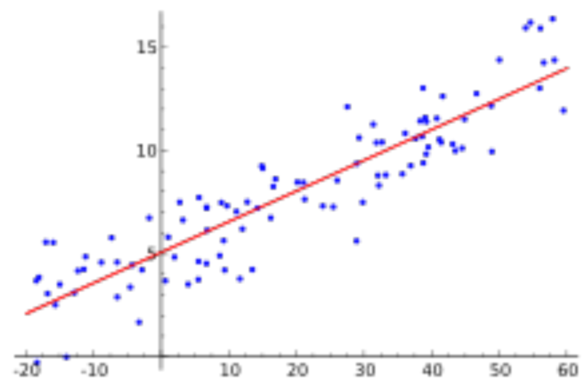
Weighted residual method

$$\int_0^2 rw = 0$$

Weighted residual



$$\int_0^2 r^2 = 0$$

Least squares

$$\int_0^2 r^2 w = 0$$

Weighted least squares

$$\int_0^2 \left(-\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x} \right) - f \right) w dx = 0$$

$$\int_0^2 -\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x} \right) w dx = \int_0^2 f w dx$$

$$\int_\Omega (\nabla^2 u) W \, d\Omega = \int_\Omega f W \, d\Omega$$

Weighted residual method

$$\int_\Omega (\nabla^2 u) W \, d\Omega = \int_\Omega f W \, d\Omega$$

$W_i = \delta(\lvert\mathbf{x} - \mathbf{x}_i\rvert)$	Collocation Method	
$W_i = \frac{\partial R}{\partial u_i}$	Least Squares Method	
$W_i = \mathbf{x}^i$	Method of Moments	
$W_i = \phi_i$	Galerkin Method	FEM
$W_i = G(\lvert\mathbf{x} - \mathbf{x}_i\rvert)$	Green's Function Solution	BEM

Weak form

$$\int_\Omega (\nabla^2 u) W \, d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad \text{integration by parts}$$

$$\int_\Omega \nabla \cdot (\nabla u W) d\Omega - \int_\Omega (\nabla u) \cdot \nabla W \, d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad \text{Gauss' divergence theorem} \qquad \int_\Omega \nabla \cdot \mathbf{u} d\Omega = \int_\Gamma \mathbf{n} \cdot \mathbf{u} d\Gamma$$

$$\int_\Gamma \mathbf{n} \cdot (\nabla u W) d\Gamma - \int_\Omega (\nabla u) \cdot \nabla W \, d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad \mathbf{n} \cdot \nabla = \partial / \partial n$$

$$\int_\Gamma \frac{\partial u}{\partial n} W \, d\Gamma - \int_\Omega (\nabla u) \cdot \nabla W \, d\Omega = \int_\Omega f W \, d\Omega$$

Integration by parts

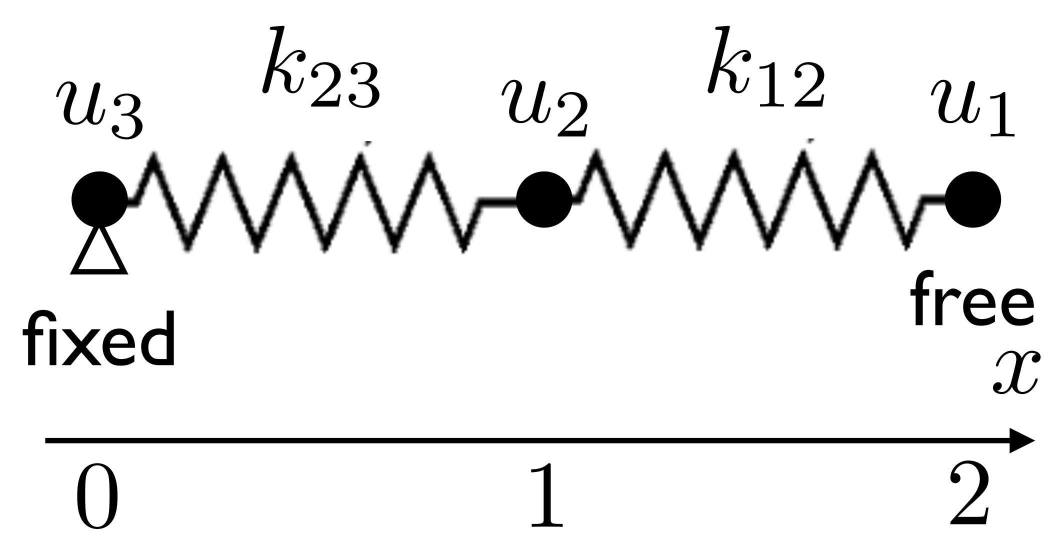
$$\int_0^2 -\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) w dx = \int_0^2 f w dx$$

$$\int_0^2 -\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) w dx = \int_0^2 c\frac{\partial \tilde{u}}{\partial x}\frac{\partial w}{\partial x} dx - \left[c\frac{\partial \tilde{u}}{\partial x} w\right]_0^2$$

Boundary conditions

$$w(0) = 0$$

$$\frac{\partial \tilde{u}(2)}{\partial x} = 0$$



Inverse form

$$\int_\Gamma \frac{\partial u}{\partial n} W \, d\Gamma - \int_\Omega (\nabla u) \cdot \nabla W \, d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad \text{integration by parts}$$

$$\int_\Gamma \frac{\partial u}{\partial n} W \, d\Gamma - \int_\Omega \nabla \cdot (u \nabla W) d\Omega + \int_\Omega u(\nabla^2 W) d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad \text{Gauss' divergence theorem}$$

$$\int_\Gamma \frac{\partial u}{\partial n} W \, d\Gamma - \int_\Gamma u \frac{\partial W}{\partial n} d\Gamma + \int_\Omega u(\nabla^2 W) d\Omega = \int_\Omega f W \, d\Omega$$

$$\Big\downarrow \quad W = G$$

$$\int_\Gamma \frac{\partial u}{\partial n} G d\Gamma - \int_\Gamma u \frac{\partial G}{\partial n} d\Gamma + \int_\Omega u(\nabla^2 G) d\Omega = \int_\Omega f G d\Omega$$

Green's function

$$\int_\Gamma \frac{\partial u}{\partial n} G d\Gamma - \int_\Gamma u \frac{\partial G}{\partial n} d\Gamma + \int_\Omega u(\nabla^2 G) d\Omega = \int_\Omega f G d\Omega$$

$$\nabla^2 G = -\delta$$

$$\int_\Omega u(\nabla^2 G) d\Omega = - \int_\Omega u \delta d\Omega$$

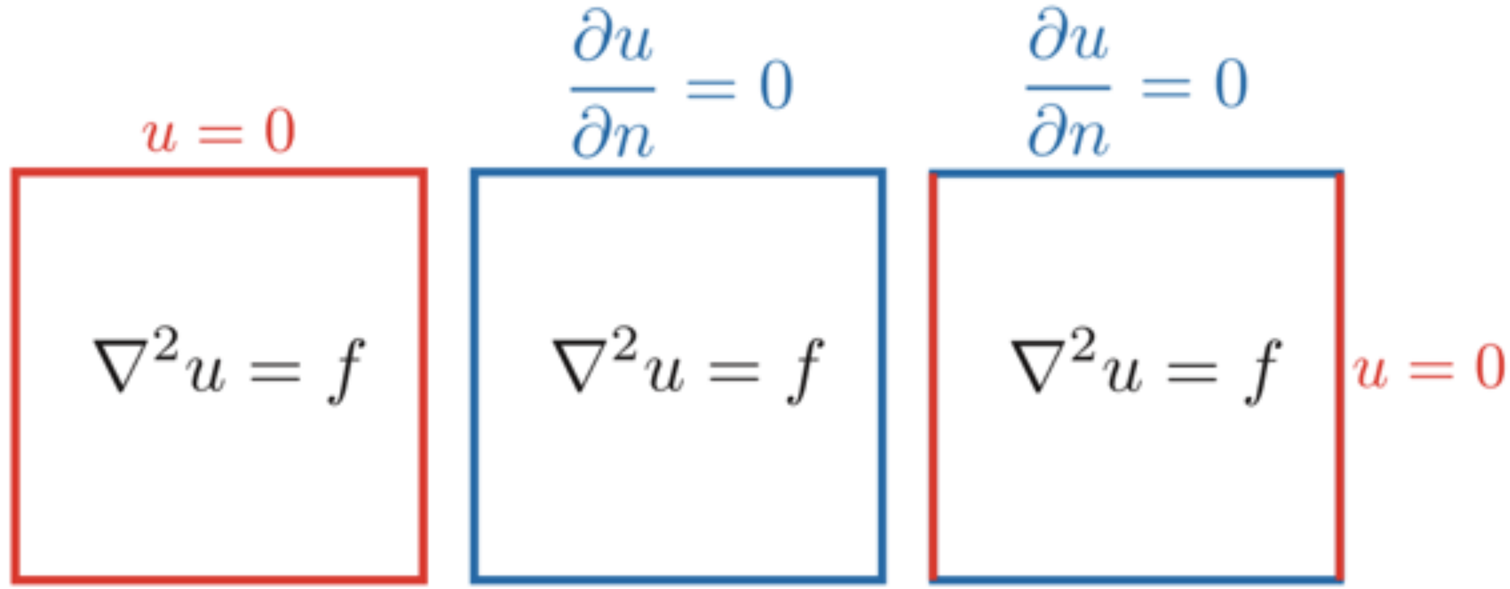
$$\int_\Gamma \frac{\partial u}{\partial n} G d\Gamma - \int_\Gamma u \left(\frac{1}{2}\delta + \frac{\partial G}{\partial n} \right) d\Gamma = \int_\Omega f G d\Omega$$

$$- \int_\Omega u \delta d\Omega \; = \; -u$$

$$- \int_\Gamma u \delta d\Gamma \; = -1/2u$$

$$N_\Gamma \left\{ \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \right. \underbrace{\begin{bmatrix} \vdots \\ \frac{\partial u_j}{\partial n} \\ \vdots \end{bmatrix}}_{\text{unknown}} = \overbrace{\begin{bmatrix} \ddots & & \\ & \frac{1}{2}\delta_{ij} + \frac{\partial G(r_{ij})}{\partial n} & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \begin{bmatrix} \vdots \\ u_j \\ \vdots \end{bmatrix} + \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Omega} \begin{bmatrix} \vdots \\ f_j \\ \vdots \end{bmatrix}$$

Boundary condition

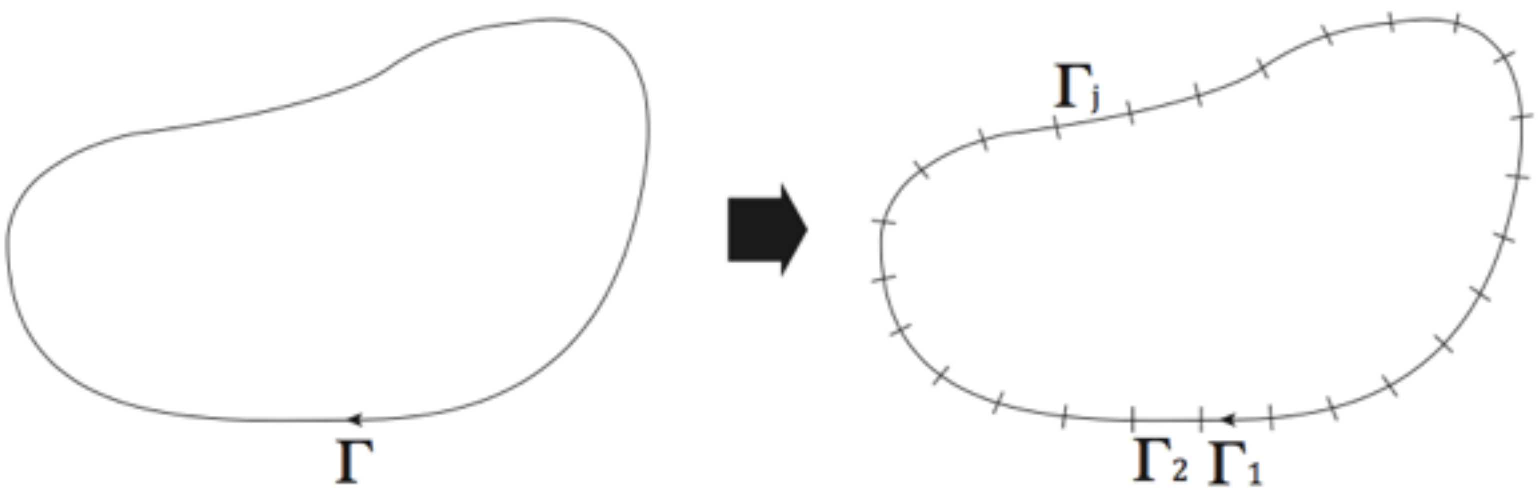


(a) Dirichlet (b) Neumann (c) Mixed

$$N_\Gamma \left\{ \overbrace{\begin{bmatrix} \ddots & & \\ & \frac{1}{2}\delta_{ij} + \frac{\partial G(r_{ij})}{\partial n} & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \right. \underbrace{\begin{bmatrix} \vdots \\ u_j \\ \vdots \end{bmatrix}}_{\text{unknown}} = \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \begin{bmatrix} \vdots \\ \frac{\partial u_j}{\partial n} \\ \vdots \end{bmatrix} - \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Omega} \begin{bmatrix} \vdots \\ f_j \\ \vdots \end{bmatrix}$$

$$N_\Gamma \left\{ \overbrace{\left[\begin{array}{c|c} \ddots & \\ G(r_{ij}) & -\frac{1}{2}\delta_{ij} - \frac{\partial G(r_{ij})}{\partial n} \\ & \ddots \end{array}\right]}^{N_\Gamma} \right. \underbrace{\left[\begin{array}{c} \vdots \\ \frac{\partial u_j}{\partial n} \\ \hline u_j \\ \vdots \end{array}\right]}_{\text{unknown}} = \overbrace{\left[\begin{array}{c|c} \ddots & \\ \frac{1}{2}\delta_{ij} + \frac{\partial G(r_{ij})}{\partial n} & -G(r_{ij}) \\ & \ddots \end{array}\right]}^{N_\Gamma} \left[\begin{array}{c} \vdots \\ u_j \\ \hline \frac{\partial u_j}{\partial n} \\ \vdots \end{array}\right] + \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Omega} \begin{bmatrix} \vdots \\ f_j \\ \vdots \end{bmatrix}$$

Discretization



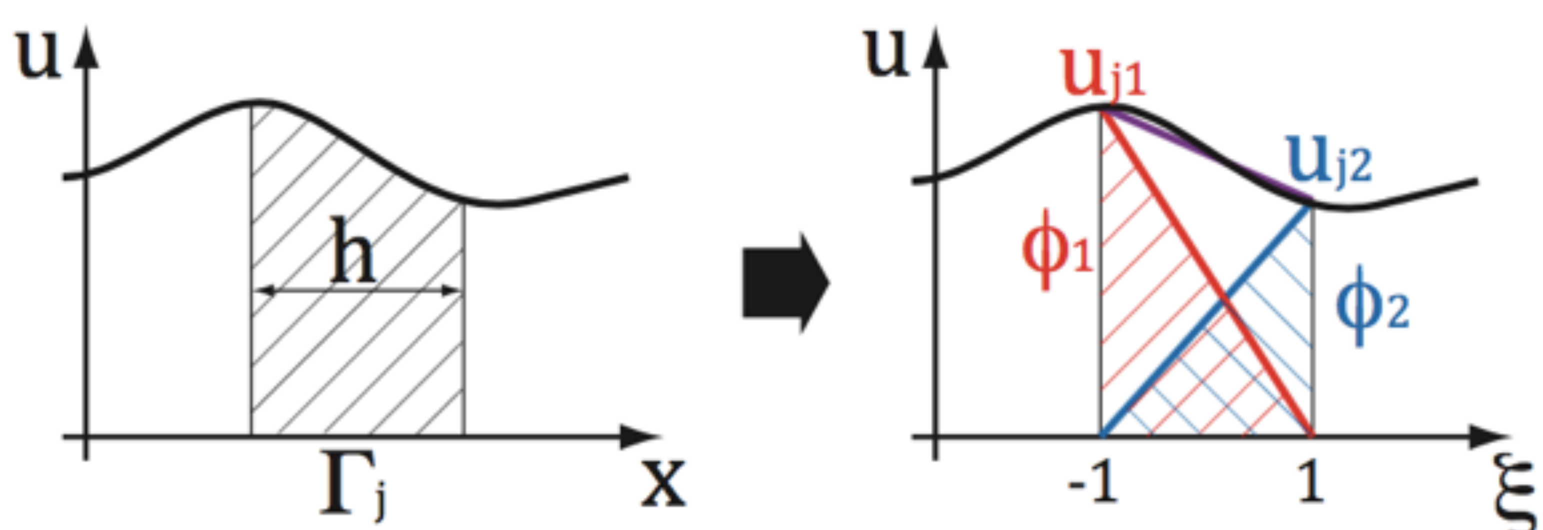
$$\int_\Gamma u \frac{\partial G}{\partial n} d\Gamma = \sum_j \int_{\Gamma_j} u \frac{\partial G}{\partial n} d\Gamma_j$$

$$\begin{aligned}
\int_{\Gamma_j} u \frac{\partial G}{\partial n} d\Gamma_j &= \int_{\Gamma_j} u_{j1}\phi_1(x) \frac{\partial G}{\partial n} d\Gamma_j + \int_{\Gamma_j} u_{j2}\phi_2(x) \frac{\partial G}{\partial n} d\Gamma_j \\
&= \int_{-1}^{1} u_{j1}\phi_1(\xi) \frac{\partial G}{\partial n} |J_j| d\xi + \int_{-1}^{1} u_{j2}\phi_2(\xi) \frac{\partial G}{\partial n} |J_j| d\xi
\end{aligned}$$

$$\phi_1(\xi) = (1-\xi)/2$$

$$\phi_2(\xi) = (1+\xi)/2$$

$$J_j = \partial \mathbf{x}_j / \partial \boldsymbol{\xi}$$



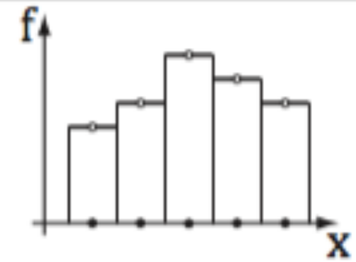
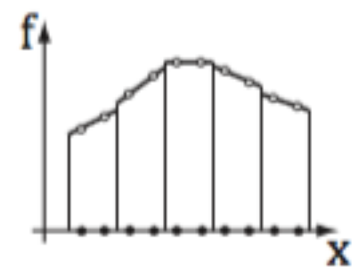
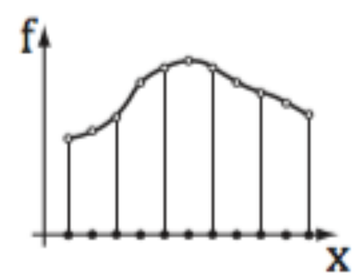
High order elements

$$= \int_{-1}^{1} u_{j1}\phi_1(\xi)\frac{\partial G}{\partial n}|J_j|d\xi + \int_{-1}^{1} u_{j2}\phi_2(\xi)\frac{\partial G}{\partial n}|J_j|d\xi$$

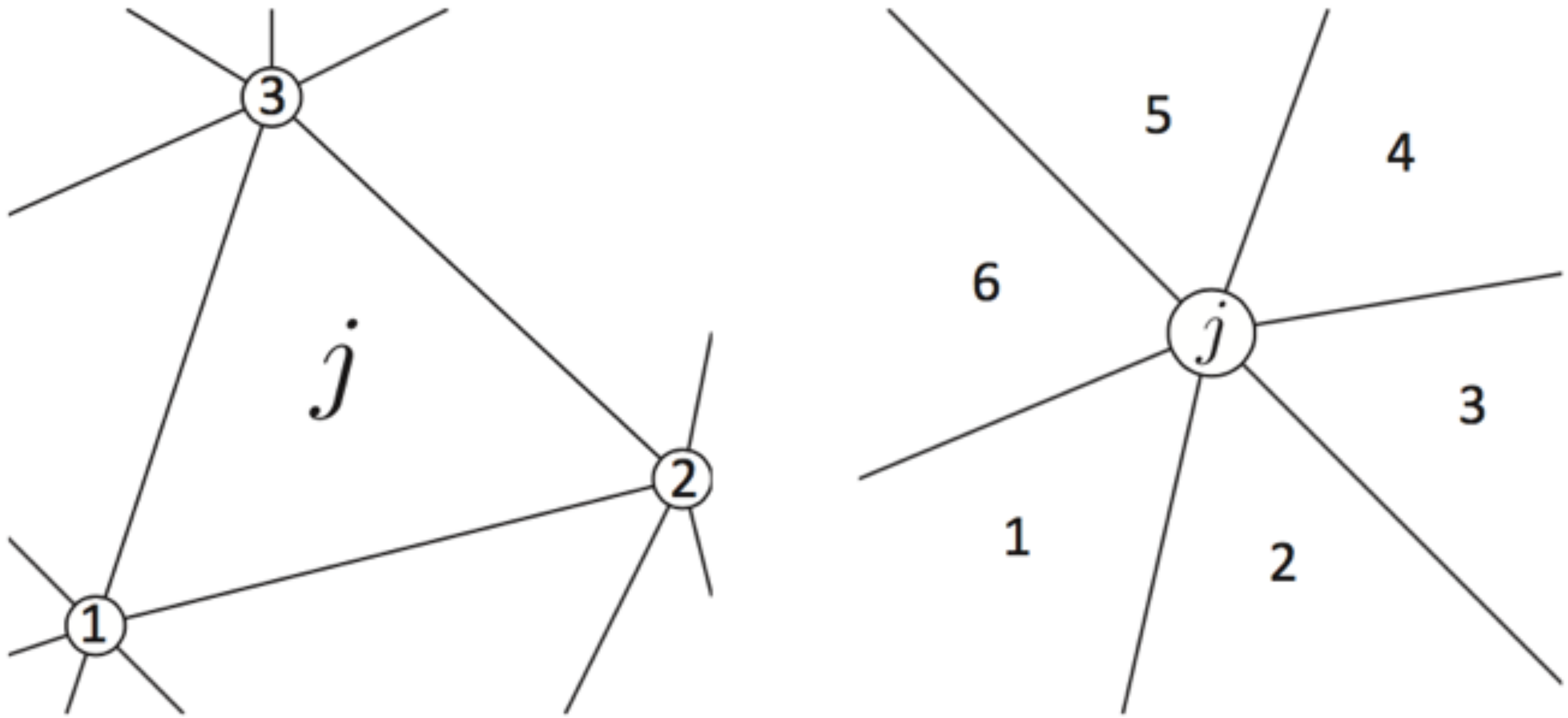
$$\int_{\Gamma_j} u\frac{\partial G}{\partial n}d\Gamma_j = |J_j|\sum_k u_{jk}\int_{-1}^{1}\phi_k(\xi)\frac{\partial G}{\partial n}d\xi$$

$$\int_{-1}^{1}\phi_k(\xi)\frac{\partial G}{\partial n}d\xi = \sum_l \phi_k(\xi_l)\frac{\partial G_{jkl}}{\partial n}w_l$$

$$\int_{\Gamma} u\frac{\partial G}{\partial n}d\Gamma = \sum_j |J_j|\sum_k u_{jk}\sum_l \phi_k(\xi_l)\frac{\partial G_{jkl}}{\partial n}w_l$$

	Order	Continous	Function
	0	no	$\phi_1 = 1$
	1	yes	$\phi_1 = (1-\xi)/2$ $\phi_2 = (1+\xi)/2$
	1	no	$\phi_1 = 1/2-\xi$ $\phi_2 = 1/2+\xi$
	2	yes	$\phi_1 = \xi(\xi-1)/2$ $\phi_2 = \xi(\xi+1)/2$ $\phi_3 = (1-\xi)(1+\xi)$

Triple loop



(a) Element centric (b) Node centeric

$$\int_\Gamma u\frac{\partial G}{\partial n}d\Gamma = \sum_j |J_j| \sum_k u_{jk} \sum_l \phi_k(\xi_l)\frac{\partial G_{jkl}}{\partial n}w_l$$

$$\int_\Gamma u\frac{\partial G}{\partial n}d\Gamma = \sum_j u_j \sum_k |J_{jk}| \sum_l \phi_{jk}(\xi_l)\frac{\partial G_{jkl}}{\partial n}w_l$$

Final matrix form

$$\int_\Gamma u \frac{\partial G}{\partial n} d\Gamma = N_\Gamma \left\{ \overbrace{\begin{bmatrix} \ddots & & \\ & \sum_k |J_{jk}| \sum_l \phi_{jk}(\xi_l) \frac{\partial G_{jkl}}{\partial n} w_l & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \right. \begin{bmatrix} \vdots \\ u_j \\ \vdots \end{bmatrix}$$

$$N_\Gamma \left\{ \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \right. \underbrace{\begin{bmatrix} \vdots \\ \frac{\partial u_j}{\partial n} \\ \vdots \end{bmatrix}}_{\text{unknown}} = \overbrace{\begin{bmatrix} \ddots & & \\ & \frac{1}{2}\delta_{ij} + \frac{\partial G(r_{ij})}{\partial n} & \\ & & \ddots \end{bmatrix}}^{N_\Gamma} \begin{bmatrix} \vdots \\ u_j \\ \vdots \end{bmatrix} + \overbrace{\begin{bmatrix} \ddots & & \\ & G(r_{ij}) & \\ & & \ddots \end{bmatrix}}^{N_\Omega} \begin{bmatrix} \vdots \\ f_j \\ \vdots \end{bmatrix}$$

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