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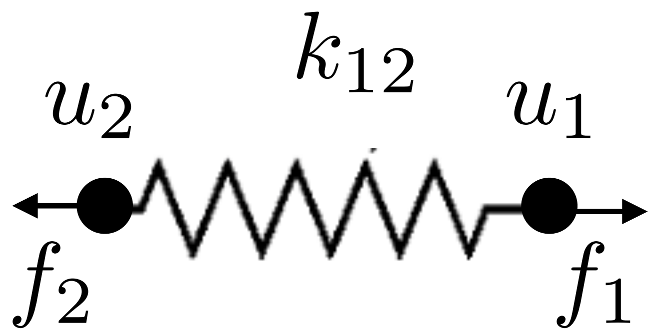
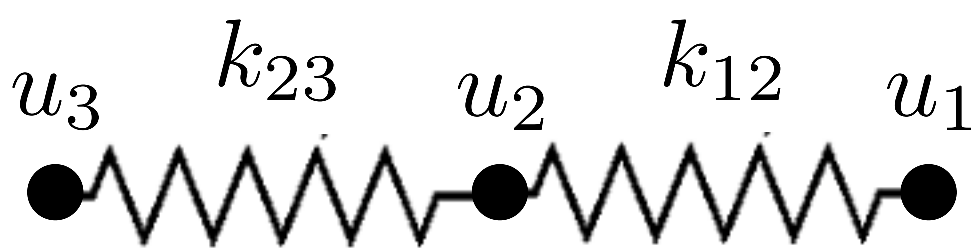
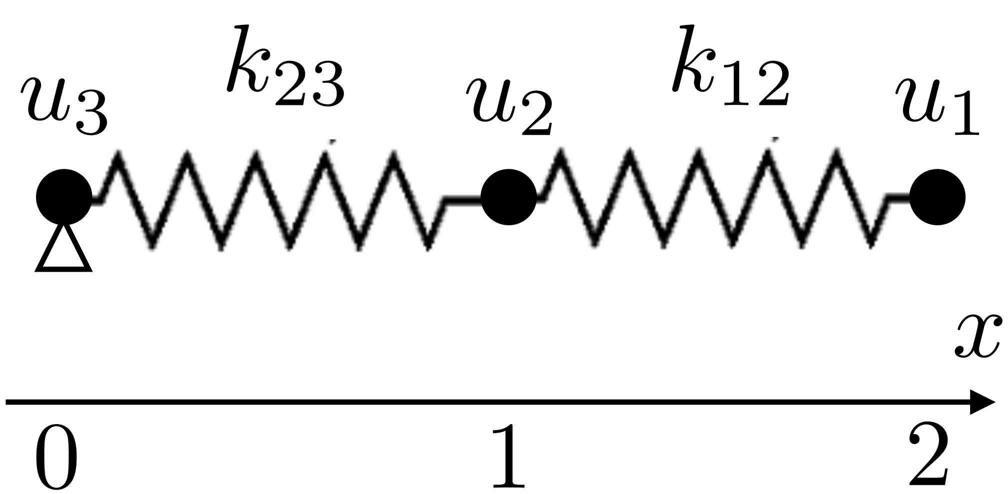
# Linear spring




$$\begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$



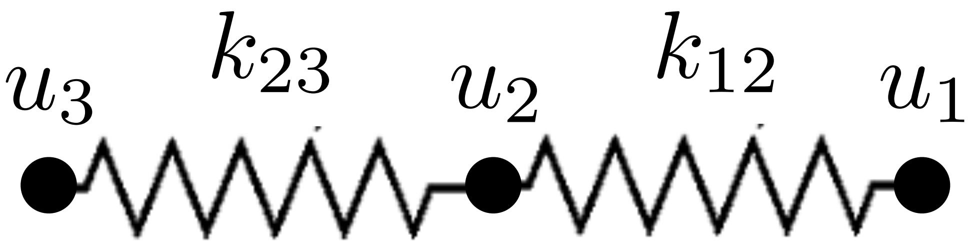

$$\begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12} + k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$




$$\begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_{23} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$-k_{23} u_2 = f_3$$

# Matrix form




$$K \quad u = f$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} k_{12} \\ k_{23} \end{matrix}$$

$$\quad u_1 \quad u_2 \quad u_3$$

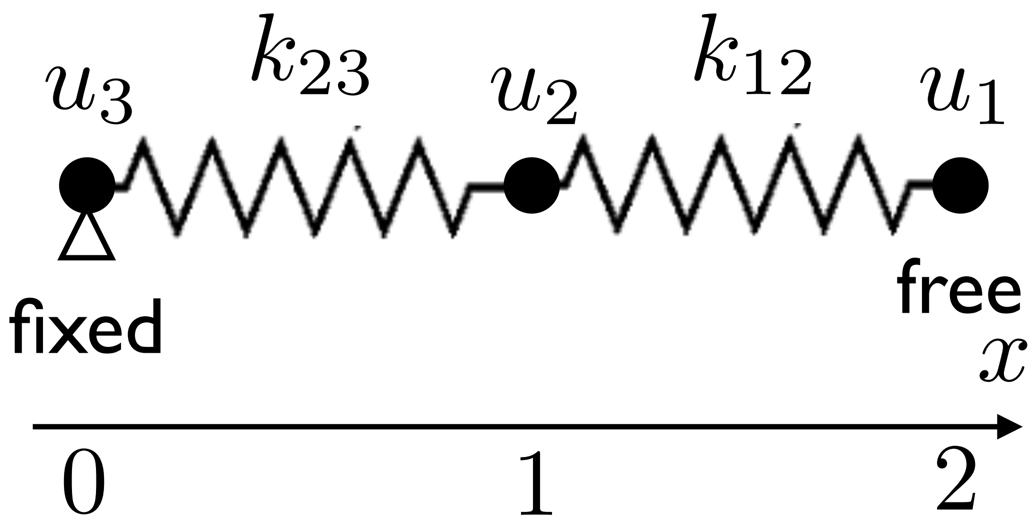
$$\begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12}+k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A^T C A = K$$

$$C = \begin{bmatrix} k_{12} & 0 \\ 0 & k_{23} \end{bmatrix}$$

# Analytical form




$$A^T C A u = f$$

$$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) = f$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$u$ : displacement

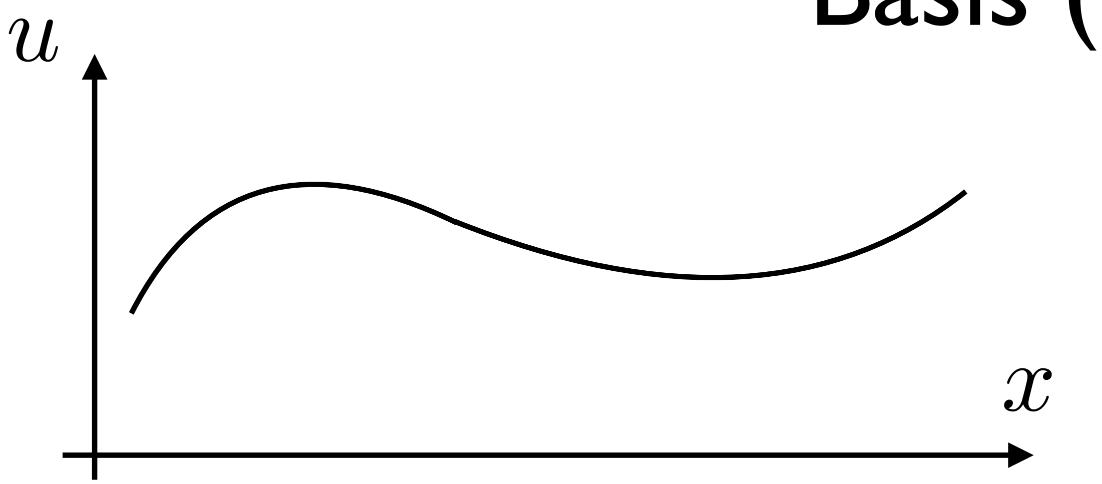
$f$ : force

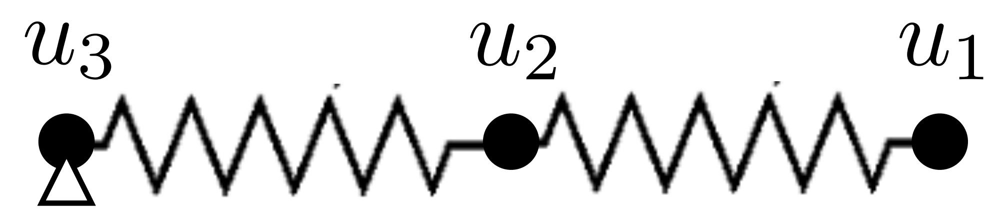
$$\epsilon = Au = \frac{\partial u}{\partial x} \longrightarrow \sigma = C\epsilon \longrightarrow f = A^T\sigma = -\frac{\partial \sigma}{\partial x}$$

$\epsilon$ : strain

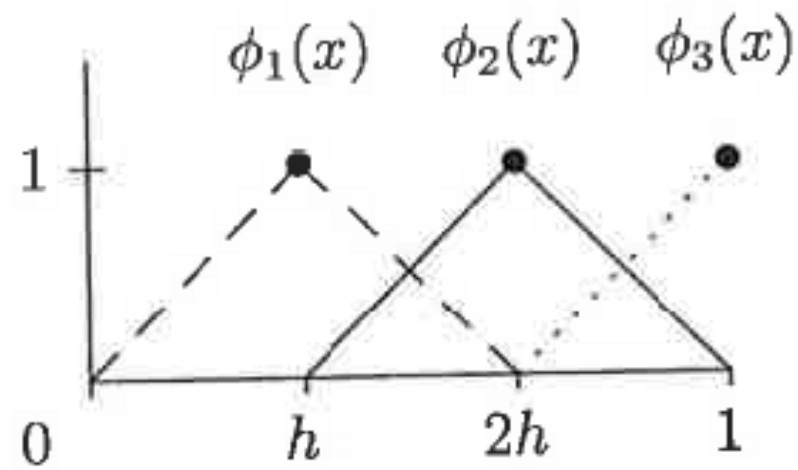
$\sigma$ : internal stress

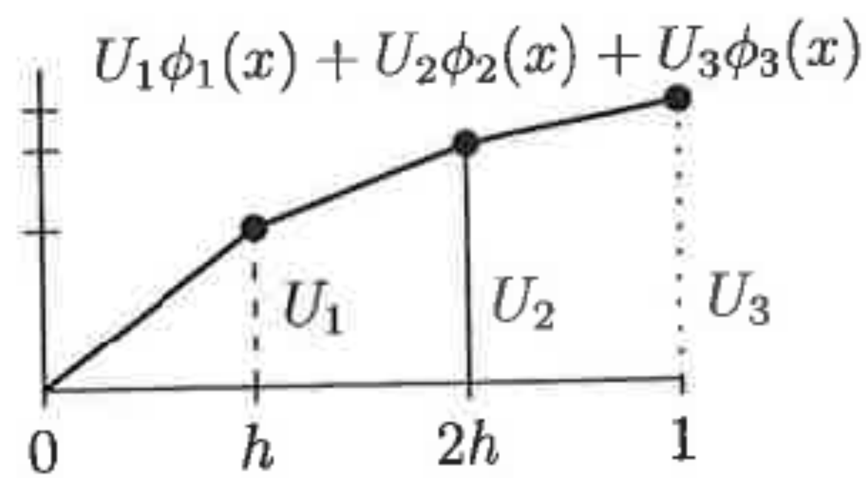
# Basis (trial) functions



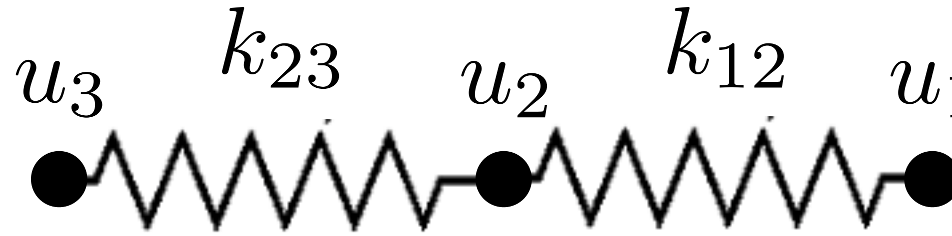

$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$



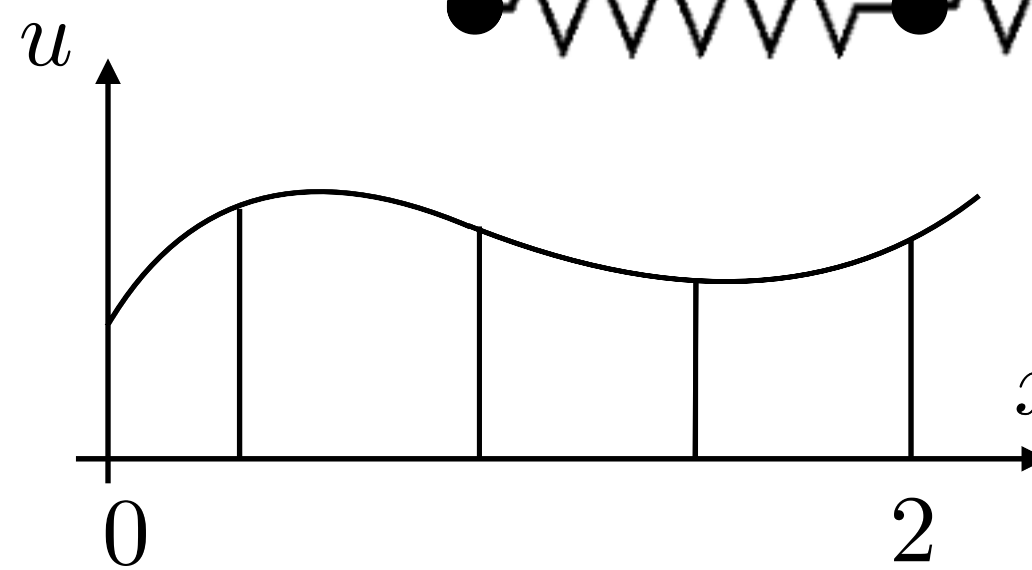




$$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) = f \quad \longrightarrow \quad \sum_{i=1}^{3} -\frac{\partial}{\partial x}\left(c\frac{\partial \phi_i}{\partial x}\right) u_i = f$$

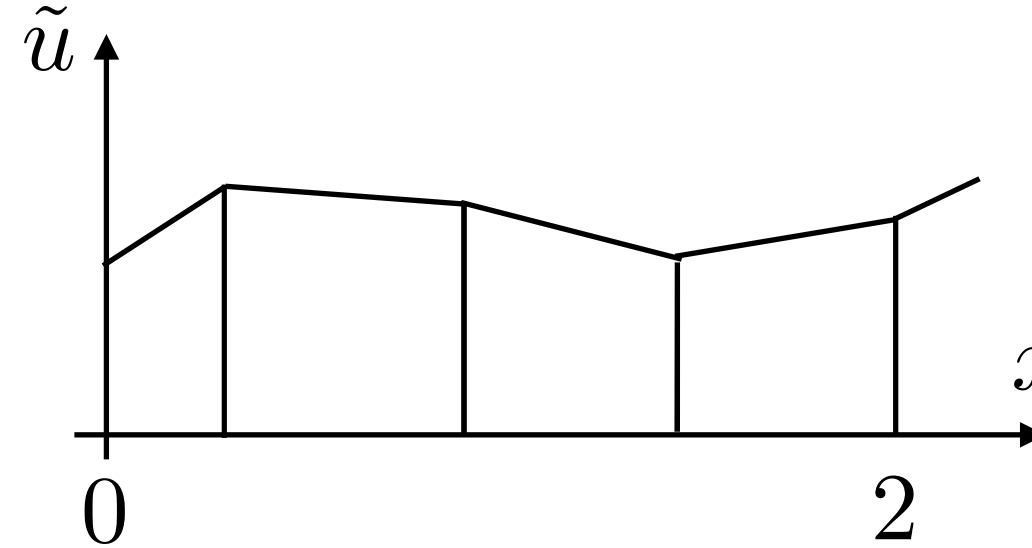
# Residual




$$-\frac{\partial}{\partial x}\left(c\frac{\partial u}{\partial x}\right) = f$$

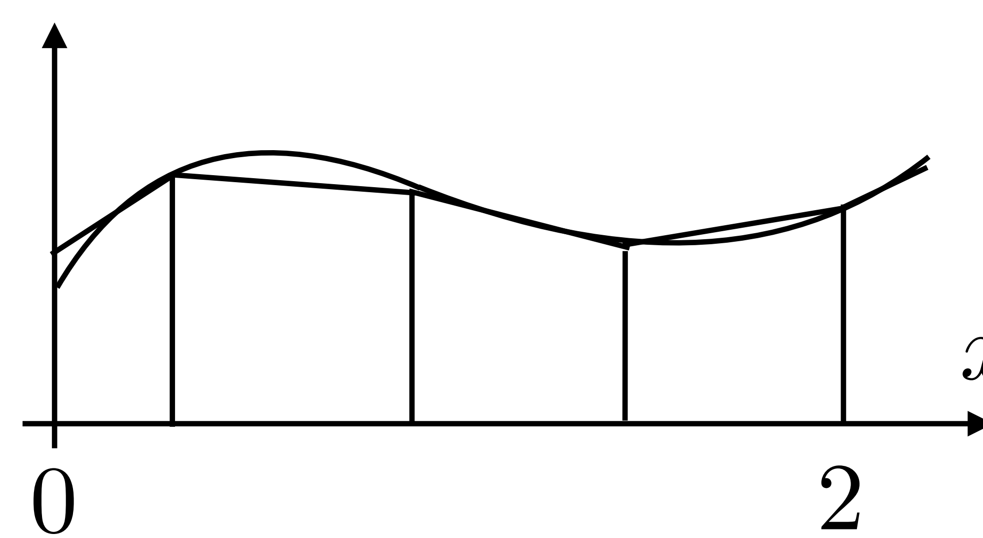



$$-\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) \approx f$$




$$-\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) - f = r$$

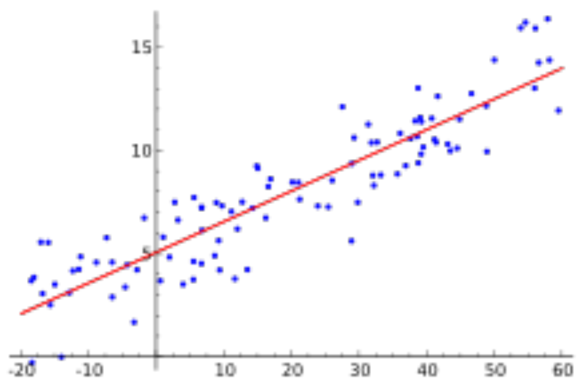
$$\int_0^2 r\,dx = 0$$



# Weighted residual method

$$\int_0^2 rw = 0$$

Weighted residual




$$\int_0^2 r^2 = 0$$

Least squares

$$\int_0^2 r^2 w = 0$$

Weighted least squares

$$\int_0^2 \left( -\frac{\partial}{\partial x} \left( c \frac{\partial \tilde{u}}{\partial x} \right) - f \right) w dx = 0$$

$$\int_0^2 -\frac{\partial}{\partial x} \left( c \frac{\partial \tilde{u}}{\partial x} \right) w dx = \int_0^2 f w dx$$

# Integration by parts

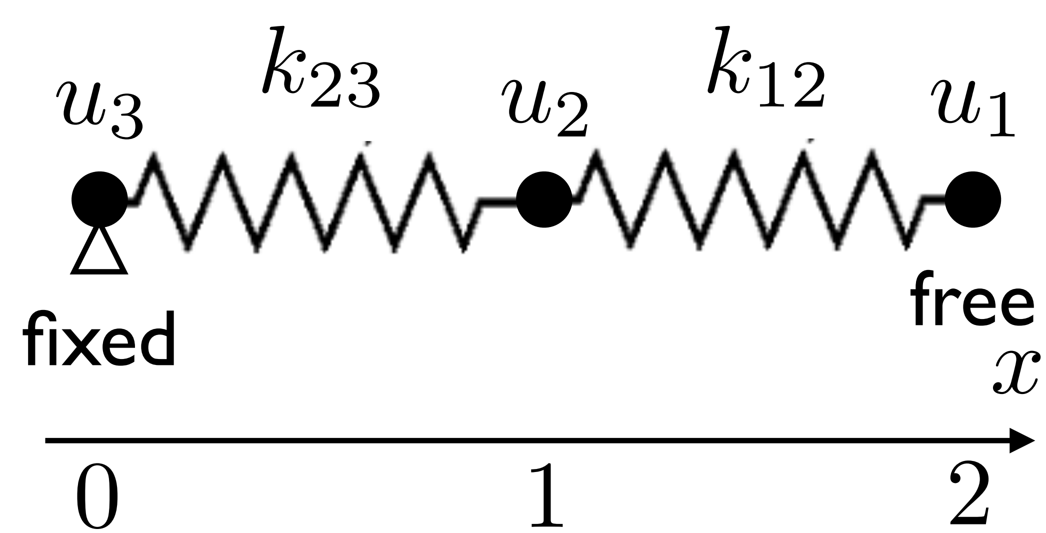
$$\int_0^2 -\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) w dx = \int_0^2 f w dx$$

$$\int_0^2 -\frac{\partial}{\partial x}\left(c\frac{\partial \tilde{u}}{\partial x}\right) w dx = \int_0^2 c\frac{\partial \tilde{u}}{\partial x}\frac{\partial w}{\partial x} dx - \left[c\frac{\partial \tilde{u}}{\partial x} w\right]_0^2$$

# Boundary conditions

$$w(0) = 0$$

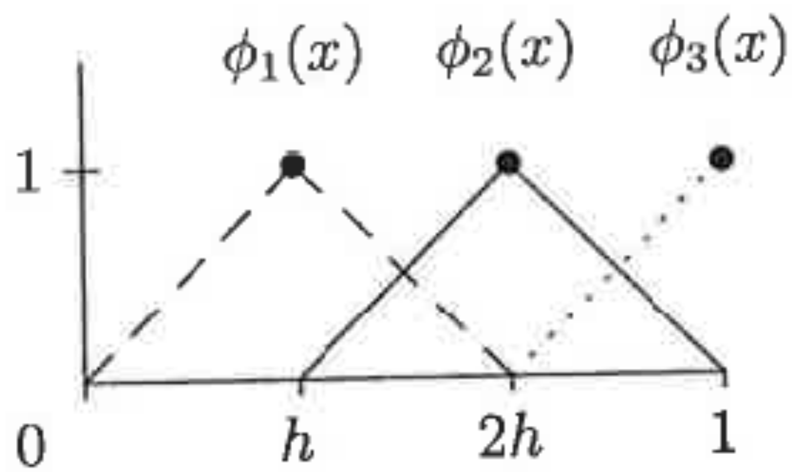
$$\frac{\partial \tilde{u}(2)}{\partial x} = 0$$

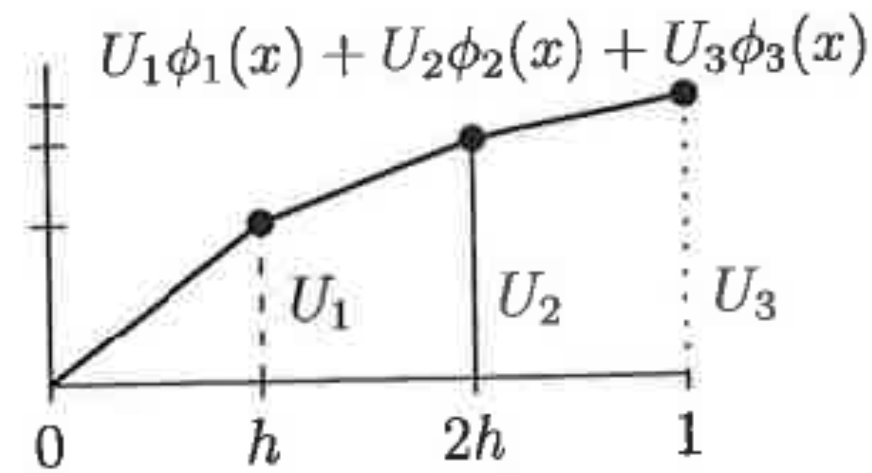


# Galerkin method

$$\int_0^2 c\frac{\partial \tilde{u}}{\partial x}\frac{\partial w}{\partial x}dx = \int_0^2 f w dx$$

$$w(x) = w_1\phi_1(x) + w_2\phi_2(x) + w_3\phi_3(x) \qquad w_1 = w_2 = w_3 = 1$$




$$w(x) = w_1(x) + w_2(x) + w_3(x)$$

$$\sum_{j=1}^{3}\int_0^2 c\frac{\partial \tilde{u}}{\partial x}\frac{\partial w_j}{\partial x}dx = \sum_{j=1}^{3}\int_0^2 f w_j dx$$

# Final discrete form

$$\sum_{j=1}^{3}\int_0^2 c\frac{\partial \tilde{u}}{\partial x}\frac{\partial w_j}{\partial x}dx = \sum_{j=1}^{3}\int_0^2 f w_j dx \qquad \frac{\partial \tilde{u}}{\partial x} = \sum_{i=1}^{3}\frac{\partial \phi_i}{\partial x}u_i$$

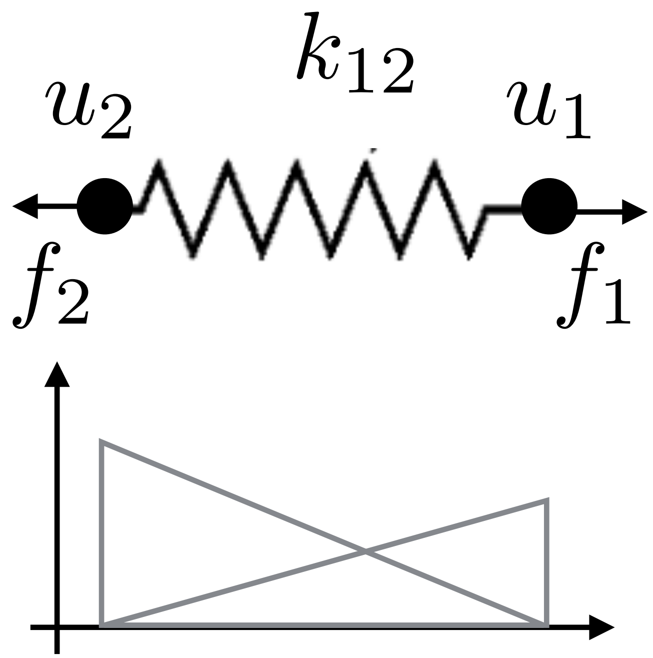
$$\sum_{i,j=1}^{3}\int_0^2 c\frac{\partial \phi_i}{\partial x}\frac{\partial w_j}{\partial x}u_i dx = \sum_{j=1}^{3}\int_0^2 f w_j dx$$

$$K_{ij} = \int_0^2 c\frac{\partial \phi_i}{\partial x}\frac{\partial w_j}{\partial x} \qquad f_j = \int_0^2 f w_j dx$$

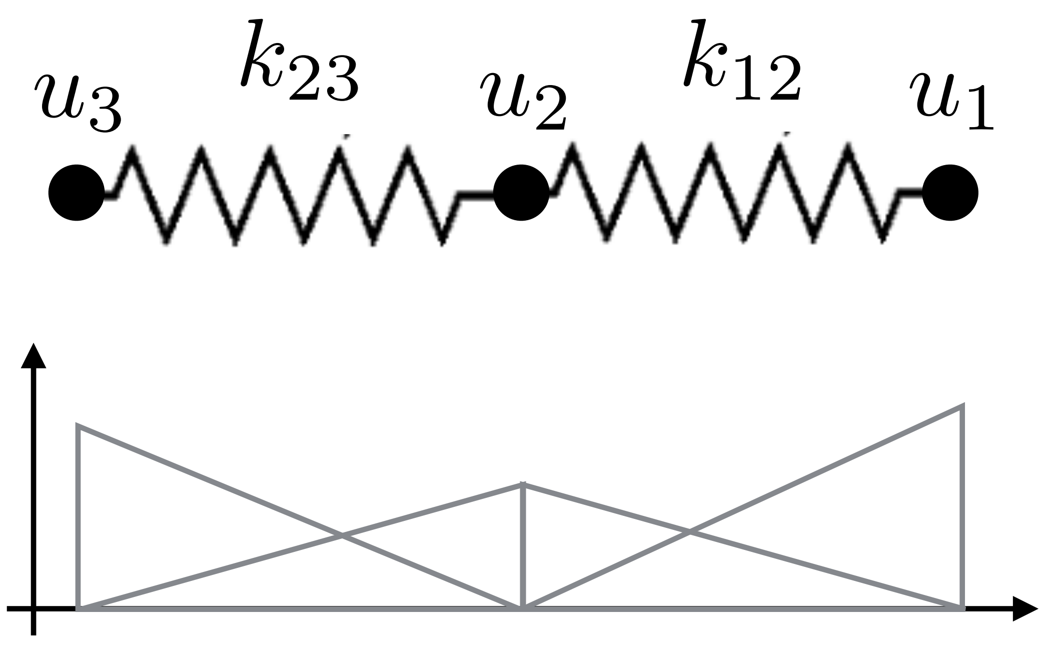
$$K \quad u = f$$

$$\begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12}+k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

# Elementwise calculation



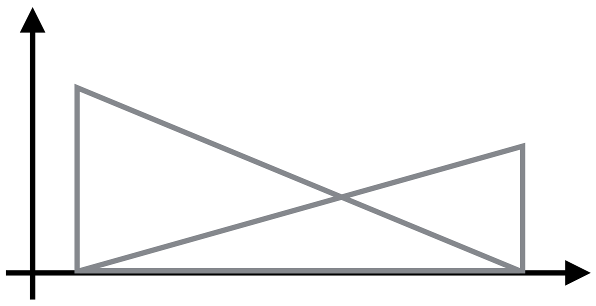

$$\begin{bmatrix} k_{12} & -k_{12} \\ -k_{12} & k_{12} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$



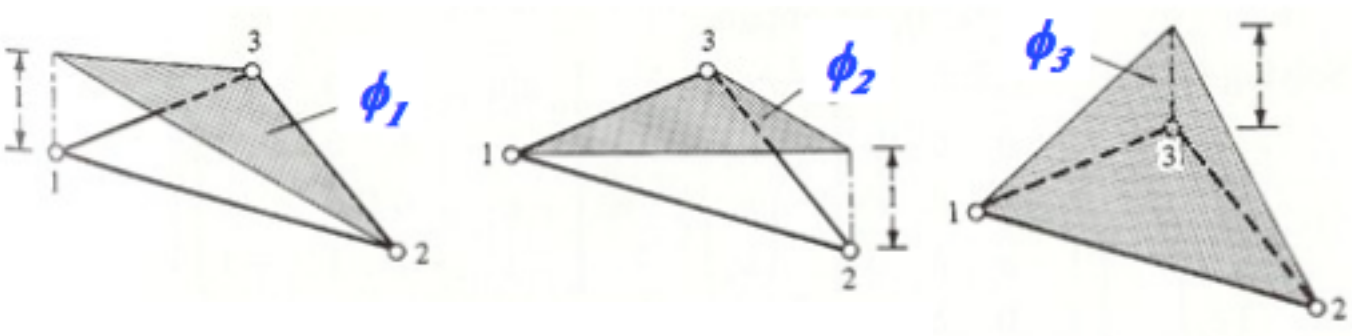

$$\begin{bmatrix} k_{12} & -k_{12} & 0 \\ -k_{12} & k_{12} + k_{23} & -k_{23} \\ 0 & -k_{23} & k_{23} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

# 2-D FEM

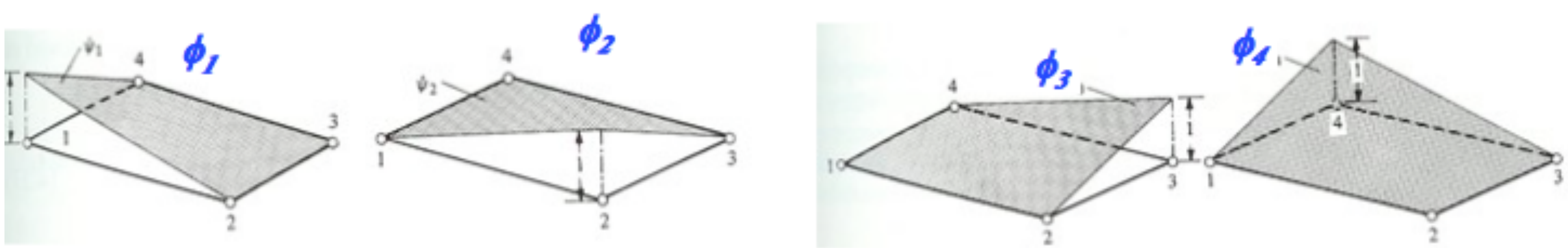
1-D



2-D triangle



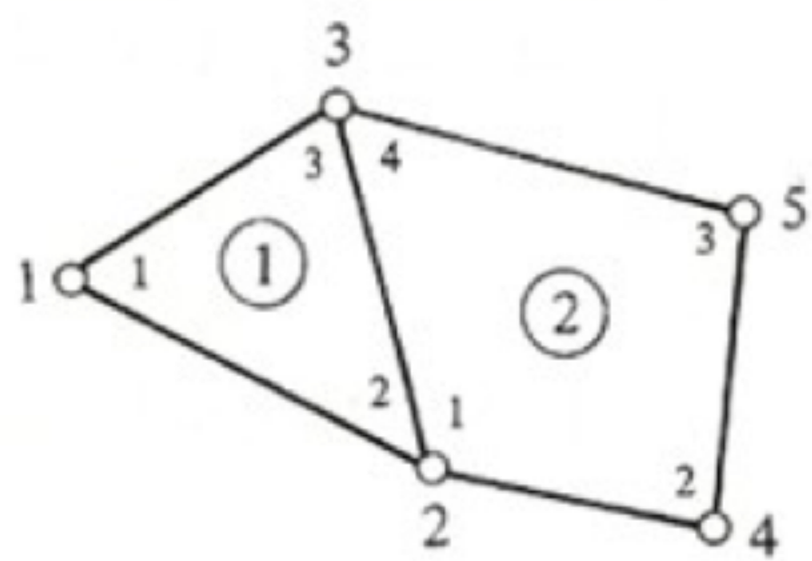

2-D rectangle



# Assembling the matrix K

$$F_i^{(e)} = \int_{\Omega_e} \phi_i Q(x,y)\,dx\,dy - \oint_{\Gamma_e} \phi_i q_n\, ds = \sum_{j=1}^{n_e} K_{ij}^{(e)} T_j^{(e)}$$

$$U_1 = T_1^{(1)},\ U_2 = T_2^{(1)} + T_1^{(2)},\ U_3 = T_3^{(1)} + T_4^{(2)},\ U_4 = T_2^{(2)},\ U_5 = T_3^{(2)},$$




| Global → | Local |
|---|---|
| $K_{11}$ | $K_{11}^{(1)}$ |
| $K_{12}$ | $K_{12}^{(1)}$ |
| $K_{22}$ | $K_{22}^{(1)} + K_{11}^{(2)}$ |
| $K_{14}$ | 0 |
| $K_{15}$ | 0 |
| $K_{23}$ | $K_{23}^{(1)} + K_{14}^{(2)}$ |

$$\begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{11}^{(2)} & K_{23}^{(1)} + K_{14}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} \\ K_{31}^{(1)} & K_{32}^{(1)} + K_{41}^{(2)} & K_{33}^{(1)} + K_{44}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} \\ 0 & K_{21}^{(2)} & K_{24}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} \\ 0 & K_{31}^{(2)} & K_{34}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \begin{pmatrix} F_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ F_3^{(1)} + F_4^{(2)} \\ F_2^{(2)} \\ F_3^{(2)} \end{pmatrix}$$

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