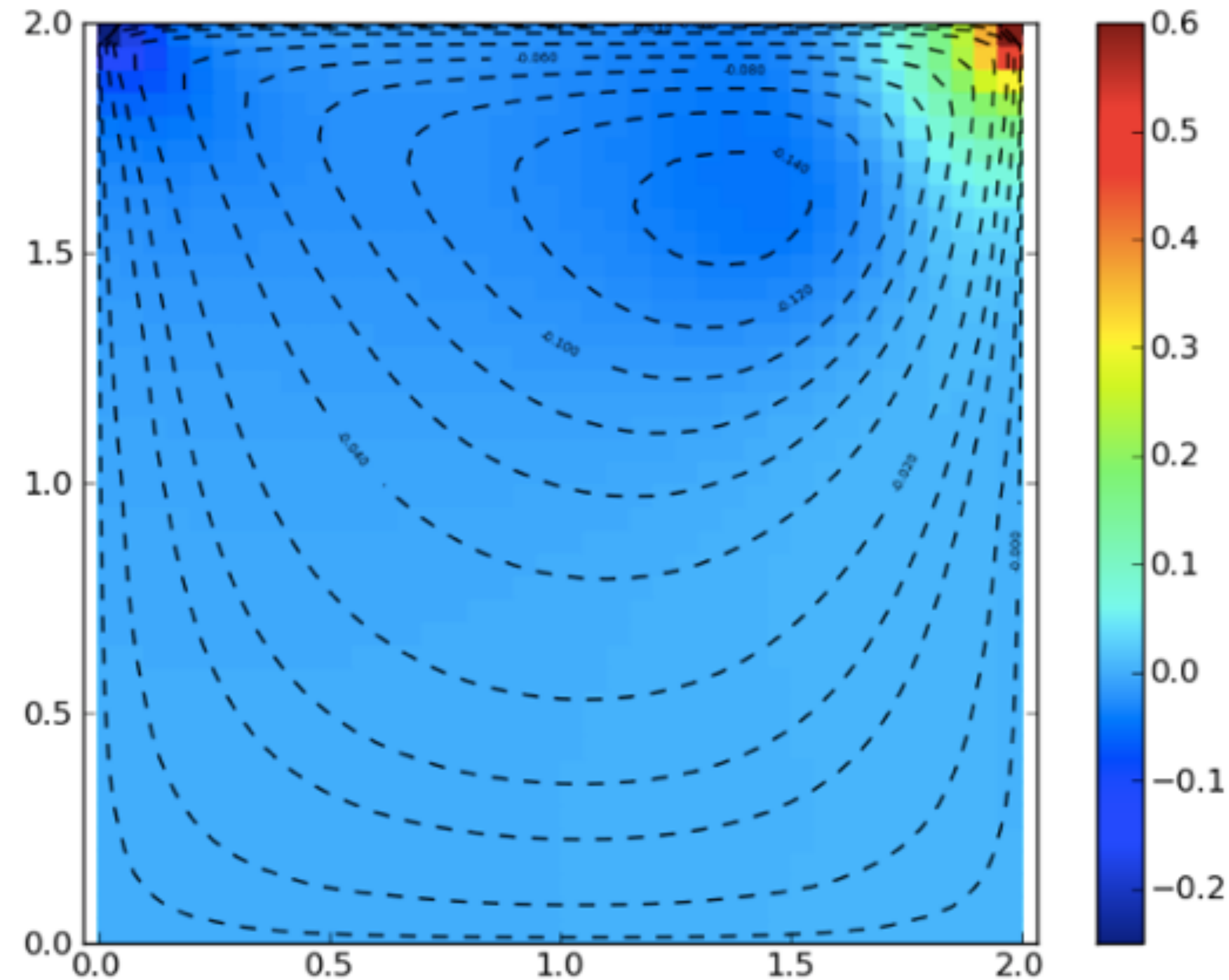
		Course schedule	Required learning
04/07	Class 1	Discretizing differential equations	Discretize differential equations using forward, backward, and central difference, with high order, and evaluate the discretization error
04/11	Class 2	Finite difference methods	Understand stability of low and high order time integration, and use it to solve convection, diffusion, and wave equations
04/14	Class 3	Finite element methods	Understand the concepts of Galerkin methods, test functions, isoparametric elements, and use it to solve elasticity equations.
04/18	Class 4	Spectral methods	Explain the advantages of orthogonal basis functions such as Fourier, Chebyshev, Legendre, and Bessel.
04/21	Class 5	Boundary element methods	Understand the relation between inverse matrices, δ functions and Green's functions, and solve boundary integral equations.
04/25	Class 6	Molecular dynamics	Understand the significance of symplectic time integrators and thermostats, and solve the dynamics of interacting molecules.
04/28	Class 7	Smooth particle hydrodynamics (SPH)	Evaluate the conservation and dissipation properties of differential operators formed from radial basis functions.
05/02	Class 8	Particle mesh methods	How to conserve higher order moments for interpolations schemes when both particle and mesh-based discretizations are used.

12 steps to the Navier-Stokes equation

1. 1-D linear convection
2. 1-D nonlinear convection
3. 1-D diffusion
4. 1-D Burgers' equation
5. 2-D linear convection
6. 2-D nonlinear convection
7. 2-D diffusion
8. 2-D Burgers' equation
9. 2-D Laplace equation
10. 2-D Poisson equation
11. 2-D cavity flow
12. 2-D channel flow



$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

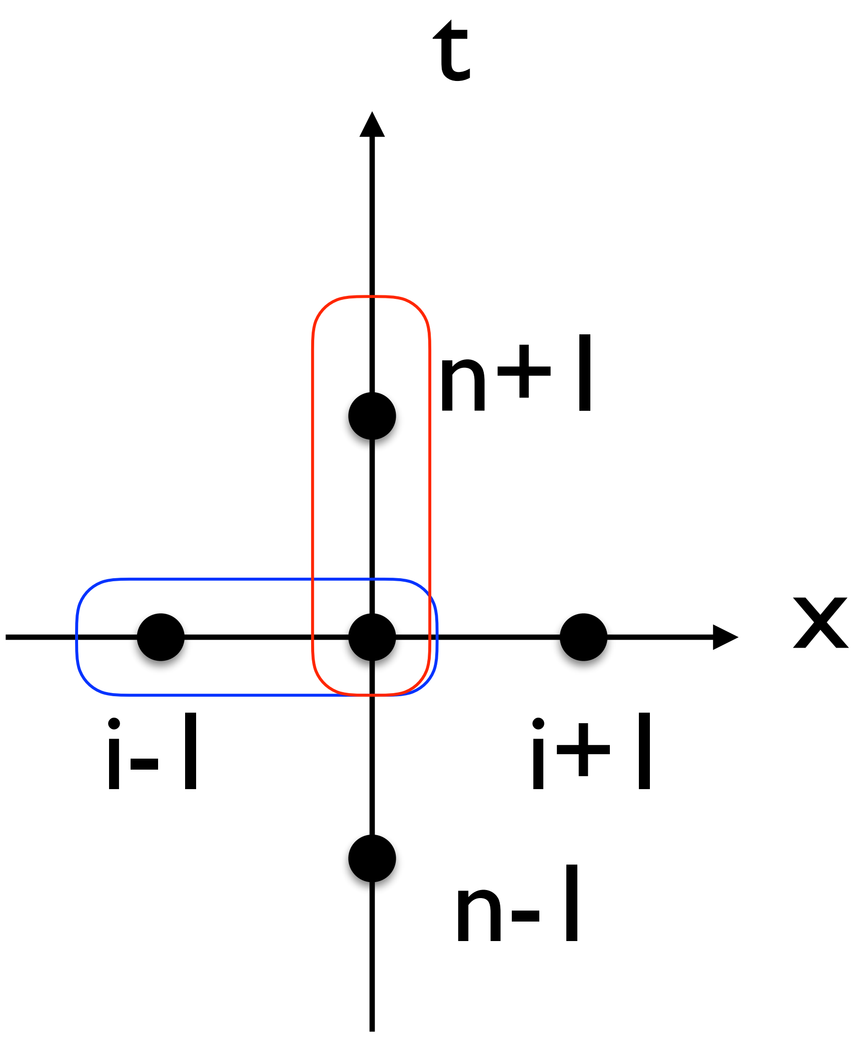
1-D linear convection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

forward backward

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$



1-D nonlinear convection

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

change c to u

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n)$$

1-D diffusion

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

second order derivative

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$u_i^{n+1} = u_i^n + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

1-D Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

convection + diffusion

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n\frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

$$u_i^{n+1} = u_i^n - u_i^n\frac{\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) + \nu\frac{\Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

2-D linear convection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$$

y term

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + c\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + c\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} = 0$$

$$u_{i,j}^{n+1} = u_{i,j}^{n} - c\frac{\Delta t}{\Delta x}(u_{i,j}^{n} - u_{i-1,j}^{n}) - c\frac{\Delta t}{\Delta y}(u_{i,j}^{n} - u_{i,j-1}^{n})$$

2-D nonlinear convection

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

y component

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} = 0$$
$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{v_{i,j}^{n} - v_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\frac{v_{i,j}^{n} - v_{i,j-1}^{n}}{\Delta y} = 0$$

$$u_{i,j}^{n+1} = u_{i,j}^{n} - u_{i,j}\frac{\Delta t}{\Delta x}(u_{i,j}^{n} - u_{i-1,j}^{n}) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}(u_{i,j}^{n} - u_{i,j-1}^{n})$$
$$v_{i,j}^{n+1} = v_{i,j}^{n} - u_{i,j}\frac{\Delta t}{\Delta x}(v_{i,j}^{n} - v_{i-1,j}^{n}) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}(v_{i,j}^{n} - v_{i,j-1}^{n})$$

2-D diffusion

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}$$

y term

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \nu \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}$$

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}) + \frac{\nu \Delta t}{\Delta y^2}(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})$$

2-D Burgers' equation

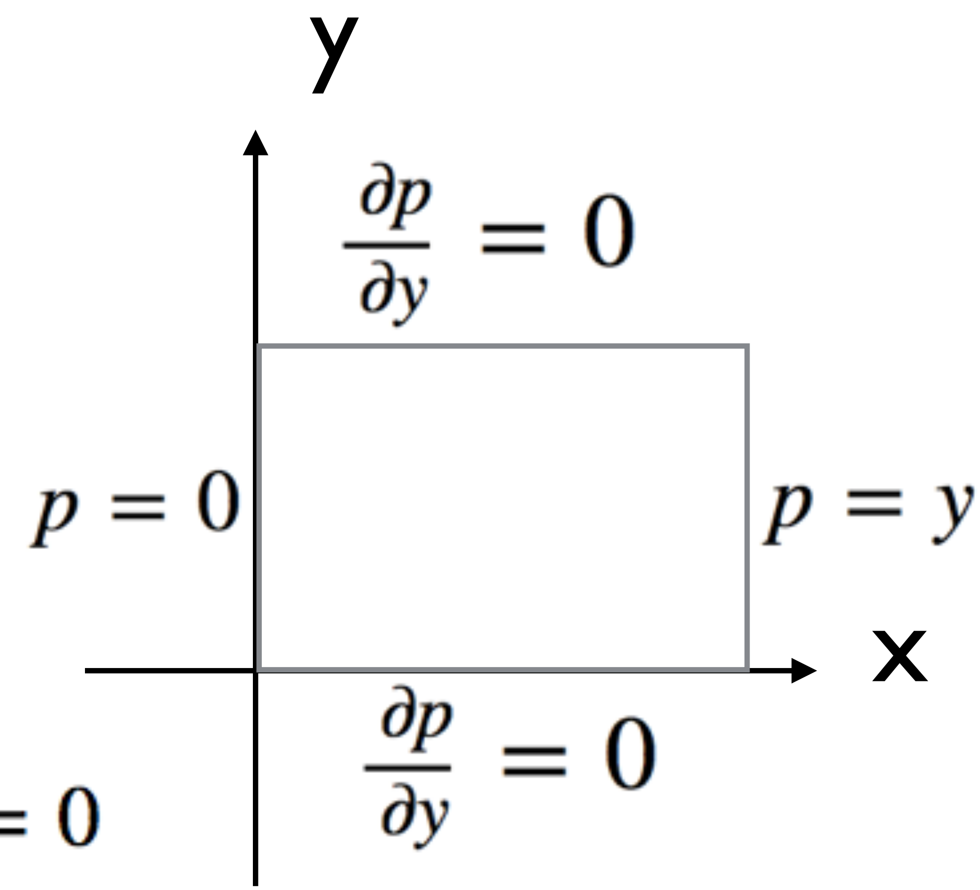
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u_{i,j}^{n+1} = u_{i,j}^n - \frac{\Delta t}{\Delta x}u_{i,j}^n(u_{i,j}^n - u_{i-1,j}^n) - \frac{\Delta t}{\Delta y}v_{i,j}^n(u_{i,j}^n - u_{i,j-1}^n)+$$
$$\frac{\nu\Delta t}{\Delta x^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) + \frac{\nu\Delta t}{\Delta y^2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j+1}^n)$$

$$v_{i,j}^{n+1} = v_{i,j}^n - \frac{\Delta t}{\Delta x}u_{i,j}^n(v_{i,j}^n - v_{i-1,j}^n) - \frac{\Delta t}{\Delta y}v_{i,j}^n(v_{i,j}^n - v_{i,j-1}^n)+$$
$$\frac{\nu\Delta t}{\Delta x^2}(v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n) + \frac{\nu\Delta t}{\Delta y^2}(v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j+1}^n)$$

2-D Laplace equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$



$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = 0$$

$$p^n_{i,j} = \frac{\Delta y^2(p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2(p^n_{i,j+1} + p^n_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}$$

2-D Poisson equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

non-zero right hand side

$$\frac{p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n}{\Delta x^2} + \frac{p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n}{\Delta y^2} = b_{i,j}^n$$

$$p_{i,j}^n = \frac{(p_{i+1,j}^n + p_{i-1,j}^n)\Delta y^2 + (p_{i,j+1}^n + p_{i,j-1}^n)\Delta x^2 - b_{i,j}^n \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

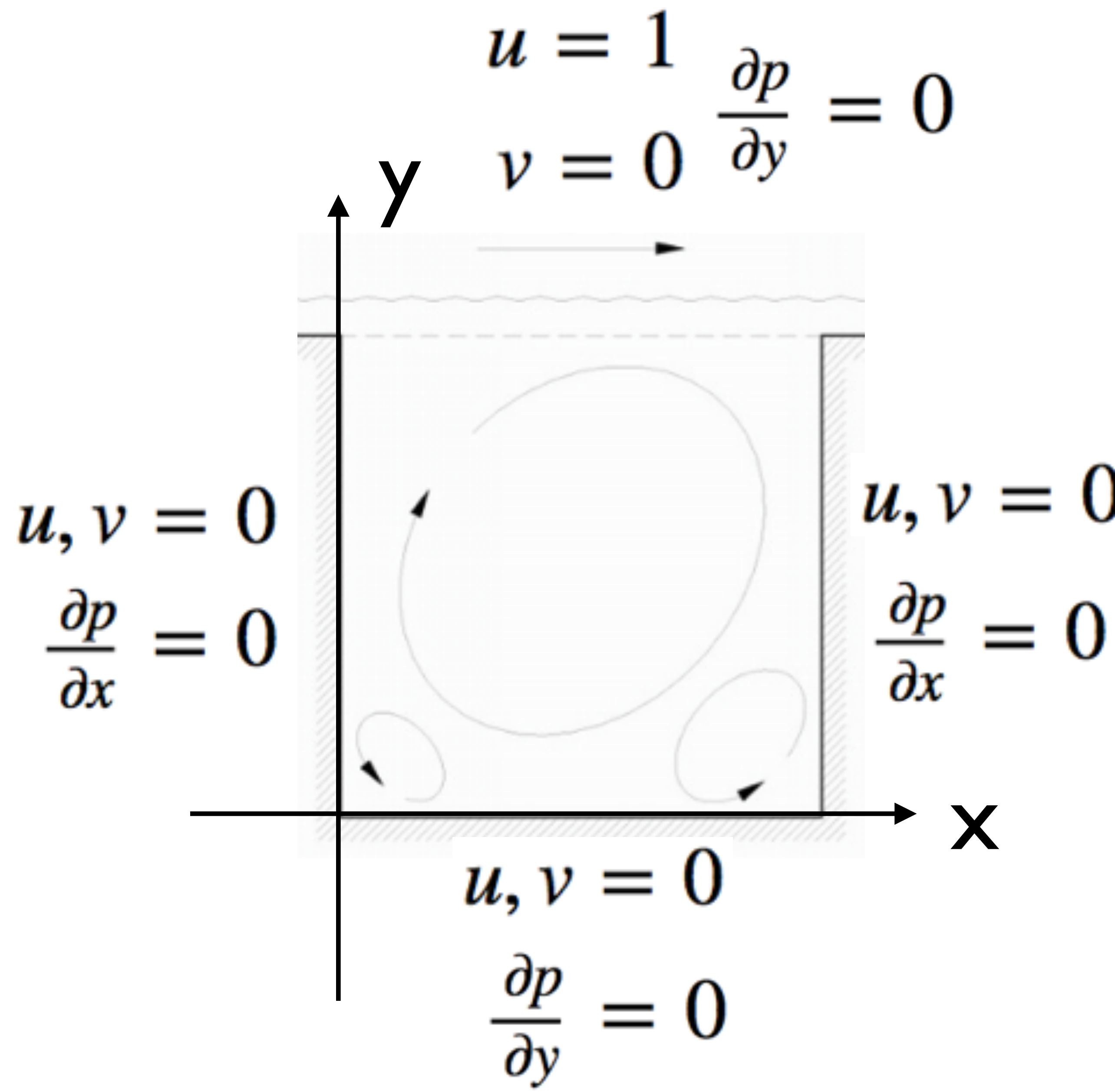
2-D cavity flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

$$u_{i,j}^{n+1} = u_{i,j}^n - u_{i,j}^n\frac{\Delta t}{\Delta x}(u_{i,j}^n - u_{i-1,j}^n) - v_{i,j}^n\frac{\Delta t}{\Delta y}(u_{i,j}^n - u_{i,j-1}^n) - \frac{\Delta t}{\rho 2\Delta x}(p_{i+1,j}^n - p_{i-1,j}^n) + \nu\left(\frac{\Delta t}{\Delta x^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) + \frac{\Delta t}{\Delta y^2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)\right)$$

$$v_{i,j}^{n+1} = v_{i,j}^n - u_{i,j}^n\frac{\Delta t}{\Delta x}(v_{i,j}^n - v_{i-1,j}^n) - v_{i,j}^n\frac{\Delta t}{\Delta y}(v_{i,j}^n - v_{i,j-1}^n) - \frac{\Delta t}{\rho 2\Delta y}(p_{i,j+1}^n - p_{i,j-1}^n) + \nu\left(\frac{\Delta t}{\Delta x^2}(v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n) + \frac{\Delta t}{\Delta y^2}(v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n)\right)$$

$$p_{i,j}^n = \frac{(p_{i+1,j}^n + p_{i-1,j}^n)\Delta y^2 + (p_{i,j+1}^n + p_{i,j-1}^n)\Delta x^2}{2(\Delta x^2 + \Delta y^2)} - \frac{\rho\Delta x^2\Delta y^2}{2(\Delta x^2 + \Delta y^2)}\times \left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) - \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - 2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right]$$



2-D channel flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

$$u_{i,j}^{n+1} = u_{i,j}^n - u_{i,j}^n\frac{\Delta t}{\Delta x}(u_{i,j}^n - u_{i-1,j}^n) - v_{i,j}^n\frac{\Delta t}{\Delta y}(u_{i,j}^n - u_{i,j-1}^n)$$
$$-\frac{\Delta t}{\rho 2\Delta x}(p_{i+1,j}^n - p_{i-1,j}^n) + \nu\left(\frac{\Delta t}{\Delta x^2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n)\right.$$
$$\left.+\frac{\Delta t}{\Delta y^2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)\right)$$

$$v_{i,j}^{n+1} = v_{i,j}^n - u_{i,j}^n\frac{\Delta t}{\Delta x}(v_{i,j}^n - v_{i-1,j}^n) - v_{i,j}^n\frac{\Delta t}{\Delta y}(v_{i,j}^n - v_{i,j-1}^n)$$
$$-\frac{\Delta t}{\rho 2\Delta y}(p_{i,j+1}^n - p_{i,j-1}^n) + \nu\left(\frac{\Delta t}{\Delta x^2}(v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n)\right.$$
$$\left.+\frac{\Delta t}{\Delta y^2}(v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n)\right)$$

$$p_{i,j}^n = \frac{(p_{i+1,j}^n + p_{i-1,j}^n)\Delta y^2 + (p_{i,j+1}^n + p_{i,j-1}^n)\Delta x^2}{2(\Delta x^2 + \Delta y^2)} - \frac{\rho\Delta x^2\Delta y^2}{2(\Delta x^2 + \Delta y^2)}\times$$
$$\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) - \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right.$$
$$\left.-2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right]$$