2. Transmission Line

2-1. Transmission-line equations

$$V(z+\Delta z) = V(z) + \frac{dV(z)}{dz}\Delta z$$

$$I(z+\Delta z) = I(z) + \frac{dI(z)}{dz}\Delta z$$

$$V(z) = (Z_d \Delta z) I(z) + V(z+\Delta z)$$

$$(Z_d = R + j\omega L)$$

$$\frac{dV(z)}{dz} = -Z_d I(z)$$

$$\frac{dI(z)}{dz} = -Y_d V(z)$$

$$(Y_d = G + j\omega C)$$

Propagation constant and characteristic impedance

$$\frac{dV(z)}{dz} = -Z_d I(z)$$

$$\frac{dI(z)}{dz} = -Y_d V(z)$$

$$\Rightarrow \quad \frac{d^2V(z)}{dz^2} = Z_d Y_d V(z)$$

$$V(z) = V_1 e^{-\gamma z} + V_2 e^{+\gamma z}$$

$$I(z) = \frac{\gamma}{Z_d}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z}) = \frac{1}{Z_c}(V_1 e^{-\gamma z} - V_2 e^{+\gamma z})$$

Important parameters

Propagation constant

$$\gamma = \sqrt{Z_d Y_d} = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta$$

Characteristic impedance

$$\frac{Z_d}{\gamma} = \sqrt{\frac{Z_d}{Y_d}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_c$$

Loss-less $\Rightarrow$

$$\gamma = j\omega\sqrt{LC} = j\beta$$

$$Z_c = \sqrt{\frac{L}{C}} = R_c$$

2-2. Reflection coefficient and input impedance

Use the coordinate y directed from load to source

$$V(y) = V_i e^{\gamma y} + V_r e^{-\gamma y}$$

$$I(y) = \frac{1}{Z_c}(V_i e^{\gamma y} - V_r e^{-\gamma y}) = I_i e^{\gamma y} + I_r e^{-\gamma y}$$

load impedance

$$Z_L = \qquad = Z_c \frac{V_i + V_r}{V_i - V_r}$$

reflection coefficient

$$S_v(0) = \qquad = \frac{Z_L - Z_c}{Z_L + Z_c}$$

matching ➡ $Z_L = Z_c$

$$S(y) = \frac{V_r e^{-\gamma y}}{V_i e^{\gamma y}} = S(0) e^{-2\gamma y}$$

input impedance

Input impedance observed at an arbitrary position y

$$Z(y) = Z_c \frac{1+S(y)}{1-S(y)} = Z_c \frac{Z_L + Z_c \tanh \gamma y}{Z_c + Z_L \tanh \gamma y}$$

In case of loss-less transmission line

$$Z(y) = R_c \frac{Z_L + jR_c \tan \beta y}{R_c + jZ_L \tan \beta y}$$

Normalization

$$z_L = \frac{Z_L}{Z_c}$$

$$S(0) = \frac{z_L - 1}{z_L + 1}$$

$$z(y) = \frac{z_L + \tanh \gamma y}{1 + z_L \tanh \gamma y}$$

2-3. Standing wave

Loss-less transmission line

$$V(y) = V_i e^{j\beta y}\left(1 + S(0)e^{-j2\beta y}\right)$$

$$I(y) = \frac{V_i e^{j\beta y}}{R_c}(1 - S(0)e^{-j2\beta y})$$

ratio of max. to min.

voltage standing wave ratio:

$$\rho = \frac{|V(y)|_{\max}}{|V(y)|_{\min}} = \frac{1 + |S(0)|}{1 - |S(0)|}$$

impedance:

at Vmax $$Z_{in} = R_{\max} = \frac{|V|_{\max}}{|I|_{\min}} = R_c \frac{1 + |S(0)|}{1 - |S(0)|} = R_c \rho$$

at Vmin $$Z_{in} = R_{\min} = \frac{|V|_{\min}}{|I|_{\max}} = R_c \frac{1}{\rho}$$

2-4. λ/4 transformer

Load impedance Z_L observed through Loss-less transmission-line

$$z(l) = \frac{z_L + j\tan\beta l}{1 + jz_L \tan\beta l}$$

$$z(l + \frac{\lambda}{4}) = \qquad = \frac{jz_L \tan\beta l + 1}{j\tan\beta l + z_L} =$$

$$z(l + \frac{\lambda}{4})z(l) =$$

$$\therefore Z(l + \frac{\lambda}{4})Z(l) =$$

λ/4 –long transmission line (R_x) is connected to R_L.

$$Z_{in}R_L = R_x^2$$

Connect this segment to another transmission line with R_c.

No reflection is obtainable if $Z_{in} = R_c$.

$$\Rightarrow R_x = \sqrt{R_c R_L}$$

2-5. Smith chart

Map a reflection coefficient on a complex plane with its corresponding impedance.

$$S = \frac{z-1}{z+1} = \frac{(r-1)+jx}{(r+1)+jx} = U + jV$$

(1) map *r* on a complex plane:

$$\left(U - \frac{r}{r+1}\right)^2 + V^2 = \left(\frac{1}{r+1}\right)^2$$

a set of circles centered at $(\frac{r}{r+1}, 0)$ with a radius of $\frac{r}{r+1}$

(2) map *x* on a complex plane:

$$(U-1)^2 + \left(V - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

a set of circles centered at $(1, \frac{1}{x})$ with a radius of $\frac{1}{|x|}$

Plot $z_L = r + jx$ ➡ $S(0)$

Rotate $S(0)$ by $-2\beta y = -2\pi(2y/\lambda)$ ➡ impedance $r' + jx'$