Starting from Maxwell's equation, we obtain set of differential equations similar to the transmission line equations.

 $rot\mathbf{E} = -j\omega\mathbf{B}$  $rot\mathbf{E} = -j\omega\mu\mathbf{H}$  $rot\mathbf{H} = j\omega\mathbf{D} + \mathbf{J}_{c} + \mathbf{J}_{s}$  $rot\mathbf{H} = j\omega\varepsilon\mathbf{E}$  $\mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}, \mathbf{J}_{c} = \sigma\mathbf{E}$  $\rho = \mathbf{J}_{s} = 0$ 

Assumption: time dependence  $exp(j\omega t)$ propagation along z direction no variation in transverse directions (x, y) no E<sub>y</sub> (E<sub>x</sub> only)

Then, Maxwell's equations are reduced to

$$0 = -j\omega\mu H_{x} \qquad -\frac{\partial H_{y}}{\partial z} = j\omega\varepsilon E_{x}$$
  
$$\frac{\partial E_{x}}{\partial z} = -j\omega\mu H_{y} \qquad \frac{\partial H_{x}}{\partial z} = j\omega\varepsilon E_{y} = 0 \qquad \Longrightarrow H_{x} = H_{z} = 0$$
  
$$0 = -j\omega\mu H_{z} \qquad 0$$

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#### Plane Wave in Free Space

| plane wave  | [ | transmission line              |                         |
|---|---|--------------------------------|-------------------------|
| $\frac{dE_x}{dz} = -j\omega\mu H_y = -(j\omega\mu' + \omega\mu'')H_y$                               |   | $\frac{dV(z)}{dz} = -Z_d I(z)$ | $(Z_d = R + j\omega L)$ |
| $\frac{dH_{y}}{dz} = -j\omega\varepsilon E_{x} = -(j\omega\varepsilon' + \omega\varepsilon'')E_{x}$ |   | $\frac{dI(z)}{dz} = -Y_d V(z)$ | $(Y_d = G + j\omega C)$ |

from the similarity to the transmission line equation

$$E_{x} = E_{x1}e^{-\gamma z} + E_{x2}e^{+\gamma z}$$

$$H_{y} = \frac{1}{Z_{c}}(E_{x1}e^{-\gamma z} - E_{x2}e^{+\gamma z}) \quad \text{with} \quad \gamma = j\omega\sqrt{\varepsilon\mu}, \quad Z_{c} = \sqrt{\frac{\mu}{\varepsilon}}$$

+z propagating wave: $E_x^+ = E_{x1}e^{j(\omega t - \beta z)}$ wave impedance:  $\frac{E_x^+}{H_y^+} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$  $H_y^+ = \frac{1}{Z_c}E_{x1}e^{j(\omega t - \beta z)}$ phase velocity:  $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}$ 

## **4.Transmission Line Composed of Two Conductors**

#### 4.1 Electro-static model

The electromagnetic (EM) wave of TEM mode can be determined by an electro-static model.

$$\mathbf{E} = \mathbf{E}_{t}(x, y)e^{-\pi}$$

$$\mathbf{H} = \mathbf{H}_{t}(x, y)e^{-\pi}$$
Use  $\nabla = \nabla_{t} + \mathbf{k}\frac{\partial}{\partial z} = \nabla_{t} - \gamma\mathbf{k}$  in  $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$ 
then,  $(\nabla_{t} - \gamma\mathbf{k}) \times \mathbf{E}_{t}e^{-\pi} = -j\omega\mu\mathbf{H}_{t}e^{-\pi}$ 

$$\nabla_{t} \times \mathbf{E}_{t} = 0 \longrightarrow \mathbf{E}_{t} = -\nabla_{t}\phi_{e}$$

$$-\gamma\mathbf{k} \times \mathbf{E}_{t} = -j\omega\mu\mathbf{H}_{t} \longrightarrow \mathbf{E}_{t} = -\nabla_{t}\phi_{m}$$

$$\nabla_{t} \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\nabla_{t} \times \mathbf{H}_{t} = 0 \longrightarrow \mathbf{E}_{t} \longrightarrow$$

#### Transmission line with two conductors

$$-\gamma \mathbf{k} \times \mathbf{E}_{t} = -j\omega\mu\mathbf{H}_{t} \longrightarrow \mathbf{H}_{t} = \frac{\gamma \mathbf{k} \times \mathbf{E}_{t}}{j\omega\mu}$$

$$div\mathbf{H} = div\mathbf{H}_{t} = \nabla \cdot \left(\frac{\gamma}{j\omega\mu}\mathbf{k} \times \mathbf{E}_{t}\right)$$
$$= \frac{\gamma}{j\omega\mu} \left(\mathbf{E}_{t} \cdot \nabla \times \mathbf{k} \cdot \mathbf{k} \cdot \nabla \times \mathbf{E}_{t}\right) = \frac{-\gamma}{j\omega\mu}\mathbf{k} \cdot \nabla \times \mathbf{E}_{t} = 0$$

$$\mathbf{E} = \mathbf{E}_{t}(x, y)e^{-\gamma z}$$
$$\mathbf{H} = \mathbf{H}_{t}(x, y)e^{-\gamma z}$$
$$\therefore \gamma^{2} = -\omega^{2}\varepsilon\mu$$

4.2 Coaxial line

charge density per unit length=q[Cm<sup>-1</sup>]

 $\rightarrow$ voltage between inner and outer conductors=V

→capacitance C[Fm<sup>-1</sup>] →characteristic impedance  $Z_c = \frac{V}{I} = \frac{\frac{q}{C}}{\frac{q}{\sqrt{\epsilon\mu}}} = \frac{\sqrt{\epsilon\mu}}{C}$ 

4.3 Strip line

Dominant guided mode is a TEM mode.

$$C = 2 \times \varepsilon \frac{w}{\frac{b}{2}} = \frac{4\varepsilon w}{b}$$
$$Z_{c} = \frac{b}{4w} \sqrt{\frac{\mu}{\varepsilon}}$$



#### Micro-strip line

4.4 Micro-strip line a quasi TEM mode approximate representation of characteristic impedance

0-th order approximation

$$C = \varepsilon \frac{w}{h} \longrightarrow Z_{c} = \sqrt{\frac{\mu}{\varepsilon} \frac{h}{w}}$$

1-st order approximation (Schneider's expression)

$$\mathcal{E}_{eff} = \frac{\mathcal{E}_r + 1}{2} + \frac{\mathcal{E}_r - 1}{2} \left( 1 + \frac{10h}{w} \right)^{-\frac{1}{2}}$$

$$\beta = \sqrt{\mathcal{E}_{eff}} \beta_0 \quad (\beta_0 = \omega \sqrt{\mathcal{E}_0 \mu_0}), \quad Z_c = \frac{1}{\sqrt{\mathcal{E}_{eff}}} Z_{c0}$$

$$Z_{c0} = 60 \ln \left( f \frac{h}{w_0} + \sqrt{1 + \left(\frac{2h}{w_0}\right)^2} \right), \quad f = 6 + 0.283 \exp \left[ -\left(30.7 \frac{h}{w_0}\right)^{0.753} \right]$$

$$w_0 = w + \Delta w = w + \frac{t}{\pi} \ln \left( 4e / \sqrt{\left(\frac{t}{h}\right)^2 + \frac{1}{\pi^2 (w/t + 1.1)^2}} \right)$$

### 5.1 Rectangular waveguide

EM field of TE and TM mode

orthogonal relation among eigen modes

$$\gamma E_{y} \equiv -j\omega\mu H_{x} \quad (5.1.a) \qquad \frac{\partial H_{z}}{\partial y} + \gamma H_{y} = j\omega\varepsilon E_{x} \quad (5.1.d)$$

$$-\gamma E_{x} = -j\omega\mu H_{y} \quad (5.1.b) \qquad -\gamma H_{x} - \frac{\partial H_{z}}{\partial x} = j\omega\varepsilon E_{y} \quad (5.1.e)$$

$$\frac{\partial E_{y}}{\partial x} - \frac{\partial E_{x}}{\partial y} = -j\omega\mu H_{z} \quad (5.1.c) \qquad \frac{\partial H_{y}}{\partial x} - \frac{\partial H_{x}}{\partial y} = 0 \qquad (5.1.f)$$

$$E_{x} = \frac{-j\omega\mu}{\gamma^{2} + \omega^{2}\varepsilon\mu} \frac{\partial H_{z}}{\partial y} = \frac{-j\omega\mu}{\beta_{c}^{2}} \frac{\partial H_{z}}{\partial y} \qquad H_{x} = \frac{-\gamma}{\beta_{c}^{2}} \frac{\partial H_{z}}{\partial x}$$

$$E_{y} = \frac{j\omega\mu}{\beta_{c}^{2}} \frac{\partial H_{z}}{\partial x} \qquad H_{y} = \frac{-\gamma}{\beta_{c}^{2}} \frac{\partial H_{z}}{\partial y}$$

### Rectangular waveguide

$$\frac{j\omega\mu}{\beta_c^2} \frac{\partial^2 H_z}{\partial x^2} + \frac{j\omega\mu}{\beta_c^2} \frac{\partial^2 H_z}{\partial y^2} = -j\omega\mu H_z$$
  
$$\therefore \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = -\beta_c^2 H_z$$
  
$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -\beta_c^2$$
  
$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -\beta_x^2$$
  
$$\frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -\beta_y^2$$
  
$$\beta_x^2 + \beta_y^2 = \beta_c^2$$

$$X(x) = A \sin \beta_x x + B \cos \beta_x x$$
$$Y(y) = C \sin \beta_y y + D \cos \beta_y y$$

#### Rectangular waveguide

$$H_{z} = X(x)Y(y)$$
  

$$X(x) = A \sin \beta_{x} x + B \cos \beta_{x} x$$
  

$$Y(y) = C \sin \beta_{y} y + D \cos \beta_{y} y$$

boundary condition at x=0, a

$$E_{y} = \frac{j\omega\mu}{\beta_{c}^{2}} \frac{\partial H_{z}}{\partial x} = \frac{j\omega\mu}{\beta_{c}^{2}} \beta_{x} (A\cos\beta_{x}x - B\sin\beta_{x}x)Y(y)$$

$$A = 0 \qquad (E_{y} = 0 \quad at \quad x = 0)$$

$$\sin\beta_{x}a = 0 \qquad (E_{y} = 0 \quad at \quad x = a)$$

$$\therefore \beta_{x} = \frac{m\pi}{a}$$

boundary condition for  $E_x$  at y=0,b

$$C = 0$$

$$\sin \beta_y b = 0 \qquad \therefore \beta_y = \frac{n\pi}{b}$$



#### Rectangular waveguide

TM mode

 $E_z = E_0 \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad (m, n \ge 1)$  $E_{x} = \frac{-\gamma}{\beta^{2}} E_{0} \frac{m\pi}{a} \cos \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$  $E_{y} = \frac{-\gamma}{\beta^{2}} E_{0} \frac{n\pi}{b} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y$  $H_{x} = \frac{j\omega\varepsilon}{R^{2}} E_{0} \frac{n\pi}{h} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{h} y$  $H_{y} = \frac{-J\omega\varepsilon}{\beta^{2}} E_{0} \frac{m\pi}{\alpha} \cos\frac{m\pi}{\alpha} x \sin\frac{n\pi}{b} y$  $\gamma = \pm \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \varepsilon \mu}$ 

## **Field distribution**



Reference: Y. Naito Micro and Millimeter Wave Engineering

#### Rectangular waveguide: orthogonality

TE mode (between transverse field component)  $E_{t}^{e} = \frac{j\omega\mu}{\beta_{c}^{2}} \left( -\frac{\partial H_{z}}{\partial y} i + \frac{\partial H_{z}}{\partial x} j \right)$   $H_{t}^{e} = \frac{-\gamma}{\beta_{c}^{2}} \left( \frac{\partial H_{z}}{\partial x} i + \frac{\partial H_{z}}{\partial y} j \right)$   $\therefore E_{t}^{e} \cdot H_{t}^{e} = \frac{-j\omega\mu\gamma}{\beta_{c}^{4}} \left( -\frac{\partial H_{z}}{\partial x} \frac{\partial H_{z}}{\partial y} + \frac{\partial H_{z}}{\partial x} \frac{\partial H_{z}}{\partial y} \right) = 0$ 

TM mode (between transverse field component)

$$E_{t}^{m} = \frac{-\gamma}{\beta_{c}^{2}} \left( \frac{\partial E_{z}}{\partial x} i + \frac{\partial E_{z}}{\partial y} j \right)$$
$$H_{t}^{m} = \frac{j\omega\varepsilon}{\beta_{c}^{2}} \left( \frac{\partial E_{z}}{\partial y} i - \frac{\partial E_{z}}{\partial x} j \right)$$
$$\therefore E_{t}^{m} \cdot H_{t}^{m} = \frac{-j\omega\varepsilon\gamma}{\beta_{c}^{4}} \left( \frac{\partial E_{z}}{\partial x} \frac{\partial E_{z}}{\partial y} - \frac{\partial E_{z}}{\partial x} \frac{\partial E_{z}}{\partial y} \right) = 0$$

## **5.2 Circular waveguide**

$$\nabla \times H = i_r \left( \frac{1}{r} \frac{\partial H_z}{\partial \theta} - \frac{\partial H_{\theta}}{\partial z} \right) + i_{\theta} \left( \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right) + i_z \frac{1}{r} \left( \frac{\partial (rH_{\theta})}{\partial r} - \frac{\partial H_r}{\partial \theta} \right)$$

$$TE \text{ mode } H_z = H_0 J_n(\beta_c r) \cos n\theta$$

$$E_r = \frac{-j\omega\mu}{\beta_c^2} \frac{1}{r} \frac{\partial H_z}{\partial \theta}$$

$$E_{\theta} = \frac{j\omega\mu}{\beta_c^2} \frac{\partial H_z}{\partial r} \longrightarrow E_{\theta} = \frac{j\omega\mu}{\beta_c^2} \beta_c H_0 J_n'(\beta_c r) \cos n\theta$$

$$H_r = \frac{-\gamma}{\beta_c^2} \frac{\partial H_z}{\partial r}$$

$$H_{\theta} = \frac{-\gamma}{\beta_c^2} \frac{1}{r} \frac{\partial H_z}{\partial \theta}$$

$$(\beta_c^2 = \gamma^2 + \omega^2 \varepsilon \mu)$$

$$J_n'(\beta_c a) = 0 \rightarrow q_{nm} = \beta_c a \quad \therefore \beta_c = \frac{q_{nm}}{a}, \quad \gamma = \pm \sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2 \varepsilon \mu}$$

### Bessel function $J_n(x)$



### Circular waveguide : TM mode

TM mode



### Bessel function $J_n(x)$



**Reference: Formula in Mathematics** 

## **Field distribution**



Reference: Y. Naito Micro and Millimeter Wave Engineering

# 5.3 Cut-off

cut-off frequency:

- The frequency at which the propagation constant of the mode concerned  $\gamma$  becomes 0.
- The mode is evanescent below that frequency.

Rectangular (a=2b): listed in the order of lower cut-off frequency

$$\begin{aligned} \mathsf{TE}_{10} \\ \mathsf{TE}_{20,} \, \mathsf{TE}_{01} \text{ (same cut-off frequency)} \qquad \gamma = \pm \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \varepsilon \mu} \\ \mathsf{TE}_{11,} \, \mathsf{TM}_{11} \\ \mathsf{TE}_{21,} \, \mathsf{TM}_{21 \, \dots \dots} \end{aligned}$$

Circular : listed in the order of lower cut-off frequency

$$TE_{11}$$

$$TE - mode: \gamma = \pm \sqrt{\left(\frac{q_{nm}}{a}\right)^2 - \omega^2 \varepsilon \mu}$$

$$TE_{21}$$

$$TE_{01}, TM_{11, \dots}$$

$$TM - mode: \gamma = \pm \sqrt{\left(\frac{p_{nm}}{a}\right)^2 - \omega^2 \varepsilon \mu}$$