Nuclear Reactor Physics Lecture Note (1)
-Perturbation Theory-

Toru Obara
Tokyo Institute of Technology

1. Perturbation theory

Perturbation theory : Effective method to calculate the change of multiplication factor by a small change in the core geometry or composition, etc.

1.1 Criticality eigenvalue problem and adjoint equation

Criticality eigenvalue problem

$$-\nabla D\nabla\phi+\Sigma_a\phi(\mathbf{r})=\frac{1}{k}\nu\Sigma_f\phi(\mathbf{r}) \qquad \cdots(1)$$

The equation can be expressed by operator notation.

$$M\phi=\frac{1}{k}F\phi \qquad \cdots(2)$$

where, $M\equiv-\nabla D(\mathbf{r})\nabla+\Sigma_a(\mathbf{r})\equiv\mathrm{Destruction\ \ operator}$ (leakage plus absorption)

$F\equiv\nu\Sigma_f(\mathbf{r})\equiv\mathrm{Production\ \ operator}$

Boundary conditions at the core surface

$$\phi(\tilde{\mathbf{r}}_s)=0 \qquad \cdots(3)$$

We define the inner product (f,g) between any two functions $f(\mathbf{r})$ and $g(\mathbf{r})$ as

$$(f,g)\equiv\int_V d^3r f^*(\mathbf{r})g(\mathbf{r}) \qquad \cdots(4)$$

where $f^*(\mathbf{r})$ denotes the complex conjugate of $f(\mathbf{r})$, an V is the core volume.

Definition of the adjoint operator $M^\dagger$:

$$\left(M^\dagger f,g\right)=(f,Mg) \qquad \cdots(5)$$

for every $f(\mathbf{r})$ and $g(\mathbf{r})$ satisfying the boundary conditions $f(\tilde{\mathbf{r}}_s)=0=g(\tilde{\mathbf{r}}_s)$.

We define the adjoint flux $\phi^\dagger$ as the solution of adjoint equation for Eq.(2), i.e.

$$M^\dagger\phi^\dagger=\frac{1}{k}F^\dagger\phi^\dagger \qquad \cdots(6)$$

1.2 First order perturbation theory

We suppose perturbation of the macroscopic absorption cross section, for example, by adding a localized absorber, to a new value

$$\Sigma_a{}'(\mathbf{r}) = \Sigma_a(\mathbf{r}) + \delta\Sigma_a(\mathbf{r}) \qquad \cdots (7)$$

We assume that the perturbation in the core absorption is small.
The corresponding change in k is governed by the perturbed criticality problem

$$M'\phi' = \frac{1}{k'}F\phi' \qquad \cdots (8)$$

The perturbation in the core absorption appears as a perturbation δM in the destruction operator

$$M' = M + \delta M, \qquad \delta M \equiv \delta\Sigma_a(\mathbf{r}) \qquad \cdots (9)$$

The scalar product of Eq.(8) with adjoint flux $\phi^\dagger$ characterizing the unperturbed core

$$\left(\phi^\dagger, M'\phi'\right) = \frac{1}{k'}\left(\phi^\dagger, F\phi'\right) \qquad \cdots (10)$$

From the definition of adjoint operator (Eq.(5))

$$\left(\phi^\dagger, M\phi'\right) = \left(M^\dagger\phi^\dagger, \phi'\right) = \left(\frac{1}{k}F^\dagger\phi^\dagger, \phi'\right) = \frac{1}{k}\left(\phi^\dagger, F\phi'\right) \qquad \cdots (11)$$

From Eq.(9), Eq.(10), Eq.(11), we find

$$\left(\frac{1}{k'} - \frac{1}{k}\right) = \frac{\left(\phi^\dagger, \delta M\phi'\right)}{\left(\phi^\dagger, F\phi'\right)} \qquad \cdots (12)$$

we can calculate $\delta k = k' - k$ from Eq.(12).
Definition of core reactivity ρ

$$\rho \equiv \frac{k-1}{k} \qquad \cdots (13)$$

The perturbation in reactivity

$$\Delta\rho = \rho' - \rho = \frac{k'-1}{k'} - \frac{k-1}{k} = \frac{1}{k} - \frac{1}{k'} \qquad \cdots (14)$$

From Eq.(9), Eq.(13), Eq.(14), we find

$$\Delta\rho = -\frac{\left(\phi^{\dagger}, \delta\Sigma_a \phi'\right)}{\left(\phi^{\dagger}, F\phi'\right)} \qquad \cdots(15)$$

If the perturbation $\delta\Sigma_a$ is small, the corresponding perturbation in the flux $\delta\phi \equiv \phi' - \phi$ is small, so

$$\Delta\rho = -\left\{\frac{(\phi^{\dagger}, \delta\Sigma_a \phi)}{(\phi^{\dagger}, F\phi)} + \frac{(\phi^{\dagger}, \delta\Sigma_a \delta\phi)}{(\phi^{\dagger}, F\phi)} - \frac{\left(\phi^{\dagger}, \delta\Sigma_a \phi\right)\left(\phi^{\dagger}, F\delta\phi\right)}{(\phi^{\dagger}, F\phi)^2} + \cdots\right\} \qquad \cdots(16)$$

Neglecting second and higher order quantities（first order perturbation theory）

$$\Delta\rho \cong -\frac{\left(\phi^{\dagger}, \delta\Sigma_a \phi\right)}{\left(\phi^{\dagger}, F\phi\right)} \qquad \cdots(17)$$

1.3 Perturbation theory in one-speed diffusion model

In one-speed diffusion model $\phi^{\dagger} = \phi$, so

$$\Delta\rho \cong \frac{\int_V d^3 r \phi(\mathbf{r}) \delta\Sigma_a(\mathbf{r}) \phi(\mathbf{r})}{\int_V d^3 r \phi(\mathbf{r}) \nu\Sigma_f(\mathbf{r}) \phi(\mathbf{r})} \qquad \cdots(18)$$

1.4 Neutron importance

We imagine an absorber inserted into the reactor core at a point $\mathbf{r}_0$ such that

$$\delta\Sigma_a(\mathbf{r}) = \alpha\delta(\mathbf{r} - \mathbf{r}_0) \qquad \cdots(19)$$

where α : effective strength of the absorber, $\delta(\mathbf{r} - \mathbf{r}_0)$: $\delta -$ function

then

$$\Delta\rho = -\frac{\int_V d^3 r \phi^{\dagger}(\mathbf{r}) \delta\Sigma_a(\mathbf{r}) \phi(\mathbf{r})}{\int_V d^3 r \phi^{\dagger}(\mathbf{r}) \nu\Sigma_f(\mathbf{r}) \phi(\mathbf{r})}$$

$$= -\frac{\alpha\phi^{\dagger}(\mathbf{r}_0)\phi(\mathbf{r}_0)}{C} \qquad C : \text{constant}$$

then

$$\phi^{\dagger}(\mathbf{r}_0) \propto \frac{\Delta\rho}{\alpha\phi(\mathbf{r}_0)}$$

$\phi^{\dagger}(\mathbf{r}_0)$ is proportional to the change in reactivity per neutron absorbed at $\mathbf{r}_0$ per unit time, so $\phi^{\dagger}(\mathbf{r}_0)$ is referred to as the neutron importance or importance function.