

Neutron Transport Theory Lecture Note (6)
- One-speed diffusion theory of a nuclear reactor (2) -

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5.2 The critical condition for general bare geometries

Considering a bare reactor of uniform composition surrounded by a free surface characterized by vacuum boundary conditions.

If the reactor is critical then the neutron flux must satisfy the steady-state diffusion equation.

$$-D\nabla^2\phi + \Sigma_a\phi(\mathbf{r}) = v\Sigma_f\phi(\mathbf{r}) \quad \dots (1)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$ $\tilde{\mathbf{r}}_s$: extrapolated boundary

Dividing Eq.(1) by $-D$,

$$\nabla^2\phi + \left(\frac{v\Sigma_f - \Sigma_a}{D}\right)\phi(\mathbf{r}) = 0 \quad \dots (2)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

$$\text{i.e.} \quad \nabla^2\phi + \left(\frac{k_\infty - 1}{L^2}\right)\phi(\mathbf{r}) = 0 \quad \dots (3)$$

boundary condition : $\phi(\tilde{\mathbf{r}}_s) = 0$

This equation is identical to that which generates the special eigenfunctions for this geometry.

$$\nabla^2\psi_n + B_n^2\psi_n(\mathbf{r}) = 0 \quad \dots (4)$$

boundary condition : $\psi(\tilde{\mathbf{r}}_s) = 0$

The requirement that the reactor is critical is the same as that for slab reactor,

$$B_m^2 \equiv \left(\frac{v\Sigma_f - \Sigma_a}{D}\right) = B_1^2 \equiv B_g^2 \quad \dots (5)$$

The critical neutron flux distribution $\phi(\mathbf{r})$ is given by the fundamental eigenfunction $\psi_1(\mathbf{r})$.

Geometric buckling and flux profile for various bare core		
Bare core geometry	Geometric buckling	Flux profile
Slab (thickness : a)	$\left(\frac{\pi}{\tilde{a}}\right)^2$	$\cos \frac{\pi x}{\tilde{a}}$
Sphere (radius : R)	$\left(\frac{\pi}{\tilde{R}}\right)^2$	$r^{-1} \sin\left(\frac{\pi r}{\tilde{R}}\right)$
Rectangular parallelepiped	$\left(\frac{\pi}{\tilde{a}}\right)^2 + \left(\frac{\pi}{\tilde{b}}\right)^2 + \left(\frac{\pi}{\tilde{c}}\right)^2$	$\cos\left(\frac{\pi x}{\tilde{a}}\right) \cos\left(\frac{\pi y}{\tilde{b}}\right) \cos\left(\frac{\pi z}{\tilde{c}}\right)$

Eq.(4) will provide us with flux shape only in critical reactor.

The magnitude of neutron flux shall be determined by the total power P generated by the core.

$$P = \int_V d^3r w_f \Sigma_f \phi(\mathbf{r}) \quad \dots (6)$$

w_f : energy produced per fission event

5.3 Reflected reactor geometries

We consider a slab reactor with reflectors of nonmultiplying material of thickness of to the both side of the core.

Time-independent diffusion equation ($x \geq 0$)

$$\text{Core : } -D^C \frac{d^2 \phi^C}{dx^2} + (\Sigma_a^C - \nu \Sigma_f^C) \phi^C(x) = 0, \quad 0 \leq x \leq \frac{a}{2} \quad \dots (7)$$

$$\text{Reflector : } -D^R \frac{d^2 \phi^R}{dx^2} + \Sigma_a^R \phi^R(x) = 0, \quad \frac{a}{2} \leq x \leq \frac{a}{2} + \tilde{b} \quad \dots (8)$$

Boundary conditions

$$\begin{aligned} (a) \quad \phi^C\left(\frac{a}{2}\right) &= \phi^R\left(\frac{a}{2}\right) \\ (b) \quad J^C\left(\frac{a}{2}\right) &= J^R\left(\frac{a}{2}\right) \\ (c) \quad \phi^R\left(\frac{a}{2} + \tilde{b}\right) &= 0 \end{aligned} \quad \dots (9)$$

General solution in the core (symmetric solution)

$$\begin{aligned}\phi^C(x) &= A^C \cos B_m^C x \\ \text{where, } B_m^C{}^2 &\equiv \frac{v\Sigma_f^C - \Sigma_a^C}{D^C}\end{aligned}\quad \dots (10)$$

Solution in reflector which satisfies boundary condition (c)

$$\begin{aligned}\phi^R(x) &= A^R \sinh \left[\frac{\frac{a}{2} + \tilde{b} - x}{L^R} \right] \\ \text{where, } L^R &= \sqrt{\frac{D^R}{\Sigma_a^R}}\end{aligned}\quad \dots (11)$$

By applying interface boundary conditions (a) and (b),

$$A^C \cos \left(\frac{B_m^C a}{2} \right) = A^R \sinh \left(\frac{\tilde{b}}{L^R} \right) \quad \dots (12)$$

$$D^C B_m^C A^C \sin \left(\frac{B_m^C a}{2} \right) = \frac{D^R}{L^R} A^R \cosh \left(\frac{\tilde{b}}{L^R} \right) \quad \dots (13)$$

Dividing Eq.(13) by Eq.(12),

$$D^C B_m^C \tan \left(\frac{B_m^C a}{2} \right) = \frac{D^R}{L^R} \coth \left(\frac{\tilde{b}}{L^R} \right) \quad \dots (14)$$

This equation is the reactor critical condition.

$$(\text{cf. } B_m^2 = B_g^2 \quad \text{in bare core})$$

Rewrite Eq.(14) as

$$\left(\frac{B_m^C a}{2} \right) \tan \left(\frac{B_m^C a}{2} \right) = \frac{D^R a}{2D^C L^R} \coth \left(\frac{\tilde{b}}{L^R} \right) \quad \dots (15)$$

$$\frac{B_m^C a}{2} < \frac{\pi}{2} \quad \text{or} \quad B_m^C{}^2 < \left(\frac{\pi}{a} \right)^2$$

$$\left[\begin{array}{c} \text{In bare (unreflected) core} \\ B_m^2 = \left(\frac{\pi}{\tilde{a}} \right)^2 \end{array} \right]$$

It is conventional to define the difference between bare and reflected core dimensions as the reflector savings δ :

$$\delta = [a(\text{bare}) - a(\text{reflected})]/2 \quad \dots (16)$$

Ex. The reflector savings for the slab core

$$\delta = \frac{1}{B_m^C} \tan^{-1} \left[\frac{D^C B_m^C L^R}{D^R} \tanh \left(\frac{\tilde{b}}{L^R} \right) \right] \quad \dots (17)$$

For the thick reflector $b \gg L^R$

$$\delta \cong \frac{D^C}{D^R} L^R \quad \dots (18)$$

5.4 Reactor criticality calculations

(1) General procedure to determine geometries and material composition of critical reactors

Diffusion equation

$$-\nabla D \nabla \phi + \Sigma_a \phi(\mathbf{r}) = v \Sigma_f \phi(\mathbf{r}) \quad \dots (19)$$

(no solutions in general unless the reactor is critical)

boundary condition

$$\phi(\tilde{\mathbf{r}}_s) = 0$$

We introduce on arbitrary parameter "k" into the equation.

$$-\nabla D \nabla \phi + \Sigma_a \phi(\mathbf{r}) = \frac{1}{k} v \Sigma_f \phi(\mathbf{r}) \quad \dots (20)$$

Picking up a core size and composition and solve the equation while determining k.

(eigenvalue problem)

k : multiplication eigenvalue

(2) Solution of eigenvalue problem by power method

Rewriting Eq.(20) in operator notation

$$M\phi = \frac{1}{k} F\phi \quad \dots (21)$$

where, $M \equiv -\nabla D(\mathbf{r}) \nabla + \Sigma_a(\mathbf{r}) \equiv$ Destruction operator (leakage plus absorption)

$F \equiv v \Sigma_f(\mathbf{r}) \equiv$ Production operator (fission)

Assuming the estimate $\phi^{(n)}$ and $k^{(n)}$ are given.

Estimate of fission source

$$S^{(n)} = F\phi^{(n)} \quad \dots (22)$$

We can iteratively solve for an improved source estimates $S^{(n+1)}$ from an earlier estimate $S^{(n)}$ by solving

$$M\phi^{(n+1)} = \frac{1}{k^{(n)}} S^{(n)} \quad \dots (23)$$

for $\phi^{(n+1)}$ and then computing

$$S^{(n+1)} = F\phi^{(n+1)} \quad \dots (24)$$

as n becomes large, $\phi^{(n+1)}$ will converge to the true eigenfunction $\phi(\mathbf{r})$ that satisfies Eq.(21) with the eigenvalue


$$M\phi^{(n+1)} \cong \frac{1}{k^{(n+1)}} F\phi^{(n+1)} \quad \dots (25)$$

If we integrate Eq.(25) overall space, we should be able to obtain a resonance estimate for $k^{(n+1)}$ as

$$k^{(n+1)} \cong \frac{\int d^3r F\phi^{(n+1)}}{\int d^3r M\phi^{(n+1)}} \quad \dots (26)$$

From Eq.(23), Eq.(24)

$$k^{(n+1)} \cong \frac{\int d^3r S^{(n+1)}(\mathbf{r})}{\frac{1}{k^{(n)}} \int d^3r S^{(n)}(\mathbf{r})} \quad \dots (27)$$


 effective fission sources
that generate $S^{(n+1)}$

This shows the eigenvalue, k in Eq.(21) is the same as the effective multiplication factor, that is the ratio of the number of neutrons in two consecutive fission generations in the reactor.

In $k \neq 1$, to make the reactor critical, we can change the reactor size and composition and repeat the calculation.