Neutron Transport Theory Lecture Note (5)
- One-speed diffusion theory of a nuclear reactor (1) -

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5. One-speed diffusion theory of a nuclear reactor

5.1 The time-dependent "slab" reactor

(a)Solution of diffusion equation

Considering a uniform slab of fissile material characterized by cross sections $\Sigma_a, \Sigma_{tr}, \Sigma_f$ (Slab reactor)

One-speed diffusion equation

$$\frac{1}{v}\frac{\partial\phi}{\partial t} - D\frac{\partial^2\phi}{\partial x^2} + \Sigma_a\phi(x,t) = \nu\Sigma_f\phi(x,t) \quad \cdots(1)$$

Initial condition

$$\phi(x,0) = \phi_0(x) = \phi_0(-x) \quad \text{(symetric)} \quad \cdots(2)$$

Boundary conditions

$$\phi\left(\frac{\tilde{a}}{2},t\right) = \phi\left(-\frac{\tilde{a}}{2},t\right) = 0 \quad \cdots(3)$$

A solution of the form (separation variables)

$$\phi(x,t) = \psi(x)T(t) \quad \cdots(4)$$

Substituting Eq.(4) to Eq.(1) and dividing by $\psi(x)T(t)$

$$\frac{1}{T}\frac{dT}{dt} = \frac{v}{\psi}\left[D\frac{d^2\psi}{dx^2} + (\nu\Sigma_f - \Sigma_a)\psi(x)\right] = \text{constant} \equiv -\lambda \quad \cdots(5)$$

hence

$$\frac{dT}{dt} = -\lambda T(t) \quad \cdots(6)$$

$$D\frac{d^2\psi}{dx^2} + (\nu\Sigma_f - \Sigma_a)\psi(x) = -\frac{\lambda}{v}\psi(x) \quad \cdots(7)$$

Solution of the time-dependent Eq.(6)

$$T(t) = T(0)e^{-\lambda t} \quad \cdots(8)$$

Space dependent equation

$$D\frac{d^2\psi}{dx^2}+\left(\frac{\lambda}{v}+\nu\Sigma_f-\Sigma_a\right)\psi(x)=0 \qquad \cdots(9)$$

Boundary condition

$$\psi\left(\frac{\tilde{a}}{2}\right)=\psi\left(-\frac{\tilde{a}}{2}\right)=0 \qquad \cdots(10)$$

here λ is still to be determined.

Considering the eigenvalue problem.

$$\frac{d^2\psi}{dx^2}+{B_m}^2\psi_n(x)$$

$$=0 \qquad \cdots(11)$$

$$\psi_n\left(\frac{\tilde{a}}{2}\right)=\psi_n\left(-\frac{\tilde{a}}{2}\right)=0$$

We are interested in symmetric solutions since $\phi_0(x)$ is symmetric.

eigen functions : $\psi_n(x)=\cos B_n x$

eigenvalue : ${B_n}^2=\left(\frac{n\pi}{\tilde{a}}\right)^2$、 $n=1,3,5,\cdots$ $\qquad \cdots(12)$

If we identify Eq.(9) as the same problem, we must choose

$$\lambda=v\Sigma_a+vD{B_n}^2-v\nu\Sigma_f\equiv\lambda_n、\quad n=1,3,5 \qquad \cdots(13)$$

λ_n : time eigenvalues

General solution of Eq.(1),

$$\phi(x,t)=\sum_{\substack{n\\odd}}A_n\exp(-\lambda_n t)\cos\frac{n\pi x}{\tilde{a}} \qquad \cdots(14)$$

The solution satisfies the boundary conditions. From initial condition Eq.(2),

$$\phi(x,0)=\phi_0(x)=\sum_{\substack{n\\odd}}A_n\cos\frac{n\pi x}{\tilde{a}} \qquad \cdots(15)$$

Using orthogonality,

$$A_n = \frac{2}{\tilde{a}}\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx\phi_0(x)\cos\frac{n\pi x}{\tilde{a}} \quad \cdots(16)$$

Thus

$$\phi(x,t) = \sum_{\substack{n\\ odd}}\left[\frac{2}{\tilde{a}}\int_{-\frac{\tilde{a}}{2}}^{\frac{\tilde{a}}{2}} dx'\phi_0(x')\cos B_n x'\right]\exp(-\lambda_n t)\cdot\cos B_n x \quad \cdots(17)$$

where the time eigenvalues λ_n are given by

$$\lambda_n = v\Sigma_a + vDB_n^{\ 2} - v\nu\Sigma_f、\quad B_n = \frac{n\pi}{\tilde{a}} \quad \cdots(18)$$

(b)Long time behavior

From Eq.(12)

$$B_1^{\ 2} < B_3^{\ 2} < \cdots < B_n^{\ 2} = \left(\frac{n\pi}{\tilde{a}}\right)^2 \quad \cdots(19)$$

hence from Eq.(18)

$$\lambda_1 < \lambda_3 < \lambda_5 \cdots \quad \cdots(20)$$

This means that the modes (terms in Eq.(17)) corresponding to larger n decay out rapidly in time.

as $t \to \infty$

$$\phi(x,t) \sim A_1 \exp(-\lambda_, t)\cos B_1 x \quad \cdots(21)$$

(fundamental mode)

This shows the regardless of the initial shape $\phi_0(x)$ the flux will decay into the fundamental mode shape.

It is usual to refer the value of $B_n^{\ 2}$ characterizing this model as

$$B_1^{\ 2} = \left(\frac{\pi}{\tilde{a}}\right)^2 \equiv B_g^{\ 2} \equiv \text{geometric buckling} \quad \cdots(22)$$

$$\left[\begin{array}{c}\text{Thus nomenclature is used since } B_n{}^2 \text{ is a} \\ \text{measure of the curvature of the mode shape} \\ B_n{}^2 = -\frac{1}{\psi_n}\frac{d^2\psi_n}{dx^2}\end{array}\right]$$

(c)Criticality condition

What is required to make the flux distribution in the reactor time-independent

i.e. what is required to make the fission chain reaction steady-state

We will define this situation to be that of reactor criticality :

Criticality ≡

when a time-independent neutron flux can be sustained in the reactor

(in the absence of sources other than fissions)

The general solution of the flux

$$\phi(x,t) = A_1 \exp(-\lambda, t) \cos B_1 x + \sum_{\substack{n=3 \\ \text{odd}}}^{\infty} A_n \exp(-\lambda_n t) \cos B_n x \qquad \cdots (23)$$

It is evident that requirement for a time-independent flux is just that the fundamental eigenvalue vanish.

$$\lambda_1 = 0 = v(\Sigma_a - \nu\Sigma_f) + vDB_1{}^2 \qquad \cdots (24)$$

since then higher modes (n=3,5,····) will have negative $-\lambda_n$ and decay out in time, leaving just,

$$\phi(x,t) \to A_1 \cos B_1 \neq \text{function of time}$$

From Eq.(24), using notation $B_1{}^2 = B_g{}^2$

$$B_m{}^2 = B_g{}^2 \qquad \text{(criticality condition)} \qquad \cdots (25)$$

$$\text{where,} \quad B_m{}^2 \equiv \frac{\nu\Sigma_f - \Sigma_a}{D} \qquad \text{(material buckling)} \qquad \cdots (26)$$

To achieve a critical reactor, we must either adjust the size ($B_g{}^2$) or the core composition ($B_m{}^2$) such that $B_m{}^2 = B_g{}^2$

we also note,

$$B_m{}^2 > B_g{}^2 \quad \Rightarrow \quad \lambda_1 < 0 \quad \Rightarrow \quad \text{super critical}$$

$$B_m{}^2 = B_g{}^2 \quad \Rightarrow \quad \lambda_1 = 0 \quad \Rightarrow \quad \text{critical}$$

$$B_m{}^2 < B_g{}^2 \quad \Rightarrow \quad \lambda_1 > 0 \quad \Rightarrow \quad \text{sub critical}$$

$$B_g{}^2 = \left(\frac{\pi}{\tilde{a}}\right)^2$$

$$B_m{}^2 = \frac{\nu\Sigma_f - \Sigma_a}{D}$$

$$t \to \infty \qquad \phi(x,t) \to A_1 \exp(-\lambda_1 t) \cdot \cos B_g x$$