### Neutron Transport Theory Lecture Note (4) -One-Speed diffusion Theory-

# Toru Obara

## Tokyo Institute of Technology

4. The one-speed diffusion equation

4.1 Derivation of the diffusion equation

We will characterize the neutron distribution in the reactor by the neutron density  $N(\mathbf{r},t)$  (or by the neutron flux  $\phi(\mathbf{r},t)$ )

We consider on arbitrary volume V of surface area S located anywhere within the reactor.

The total number of neutron in V at time t

$$\int_{V} d^{3}r N(\mathbf{r},t) = \int_{V} d^{3}r \frac{1}{v} \phi(\mathbf{r},t) \qquad \cdots (1)$$

The time rate of change of the number of neutron in V

$$\frac{d}{dt} \left[ \int_{V} d^{3}r \frac{1}{\nu} \phi(\mathbf{r}, t) \right] = \int_{V} d^{3}r \frac{1}{\nu} \frac{\partial \phi}{\partial t}$$
  
= Production in V-Absorption in V-Net leakage from V ....(2)

Production in V=  $\int_{V} d^{3}r S(\mathbf{r},t)$ 

...(3)

 $S(\mathbf{r},t)$  : neutron source density

Asorption in 
$$V = \int_V d^3 r \Sigma_a(\mathbf{r}) \phi(\mathbf{r}, t)$$
 ... (4)

Net leakage from 
$$V = \int_{S} d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t)$$
 ... (5)

Convert the surface integral into volume integral by using Gauss's theory

$$\int_{S} d\mathbf{S} \cdot \mathbf{J}(\mathbf{r}, t) = \int_{V} d^{3}r \nabla \cdot \mathbf{J}(\mathbf{r}, t) \qquad \cdots (6)$$

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = \operatorname{div} \mathbf{J}(\mathbf{r},t)$$

Substituting each of these mathematical expressions into Eq.(2),

$$\int_{\mathbf{V}} d^{3}r \left[ \frac{1}{v} \frac{\partial \Phi}{\partial t} - \mathbf{S} + \Sigma_{a} \Phi + \nabla \cdot \mathbf{J} \right] = 0 \qquad \cdots (7)$$

Eq.(7) must be hold for any volume V

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} - S + \Sigma_{a}\Phi + \nabla \cdot \mathbf{J} = 0 \qquad \cdots (8)$$

The equation contains two unknowns  $\phi(\mathbf{r},t)$  and  $J(\mathbf{r},t)$ . There is no exact relationship between  $\phi(\mathbf{r}, t)$  and  $J(\mathbf{r}, t)$ .

Diffusion approximation

$$\mathbf{J}(\mathbf{r},t) \cong -\mathbf{D}(\mathbf{r})\nabla\phi(\mathbf{r},t) \qquad \cdots (9)$$

where

$$D = \frac{1}{3\Sigma_{tr}} = \frac{1}{3(\Sigma_t - \overline{\mu}_0 \Sigma_s)}$$
 (Diffusion coefficient) .... (10)

 $\overline{\mu}_0$  : The average cosine of the scattering angle in a neutron scattering collision

Substituting Eq.(9) into Eq.(8),

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} = \nabla \cdot \mathbf{D}(\mathbf{r})\nabla \phi(\mathbf{r}) - \Sigma_{a}(\mathbf{r})\phi(\mathbf{r},t) + \mathbf{S}(\mathbf{r},t) \qquad \cdots (11)$$

(One-speed neutron diffusion equation)

If the D and  $\Sigma_a\,$  do not depend on position  $\,$  (homogeneous)

$$\frac{1}{v}\frac{\partial \Phi}{\partial t} - D\nabla^2 \Phi + \Sigma_a \Phi(\mathbf{r}, t) = S(\mathbf{r}, t) \qquad \cdots (12)$$

If the flux is not function of time (steady state)

$$-D\nabla^2 \phi + \Sigma_a \phi(\mathbf{r}) = S(\mathbf{r})$$
...(13)

**a** .

Dividing by -D

$$\nabla^2 \phi(\mathbf{r}) - \frac{1}{L^2} \phi(\mathbf{r}) = -\frac{S(\mathbf{r})}{D} \qquad \cdots (14)$$

where

$$L \equiv \sqrt{\frac{D}{\Sigma_a}} \qquad (Diffusion \ length)$$

L : A measure of how far the neutrons will diffuse from a source before they are absorbed.

[Important]

- The conditions that the diffusion approximation can be valid
  - ① It is used to describe the neutron flux several mean free path away from the boundaries or isolated source
  - <sup>(2)</sup> The medium is only weakly absorbing
  - ③ The neutron current is changing slowly on a time scale comparable to the mean time between neutron-nuclei collision

### 4.2 Initial and Boundary Conditions

(1)Initial condition

Specifying the neutron flux  $\phi(\mathbf{r}, 0)$  for all positions  $\mathbf{r}$  at the initial time t=0 Initial condition :  $\phi(\mathbf{r}, 0) = \phi_0(\mathbf{r})$ 

### (2)Boundary conditions

(a)Vacuum boundary

Ex. Outside boundary of a reactor

No neutrons can enter the reactor through this surface from outside.

A boundary condition that give correct neutron flux deep within the reactor where diffusion theory is valid is :

$$\begin{split} \varphi(\widetilde{x_s}) &= 0 \\ \widetilde{x_s} &= x_s + z_0 \quad (extrapolated \ boundary) \\ where \\ z_0 &= 0.7104 \lambda_{tr} \quad (extrapolation \ length) \end{split}$$

$$\lambda_{tr} = \frac{1}{\Sigma_{tr}}$$

(b)Boundary condition for interfaces (material discontinuity)

① Continuity of the neutron flux  $\phi(\mathbf{r}, t)$ 

(2) Continuity of the normal component of the neutron current density  $\mathbf{J}(\mathbf{r},t)$ 

(c)The condition which should be satisfied always

 $0 \le \phi(\mathbf{r}, t) < \infty$  (except in the neighborhood of localized sources)

4.3 Neutron diffusion in nonmultiplying media

Example: Neutron flux  $\varphi(x)$  in an infinite homogeneous medium with plane source at the origin

Diffusion equation

$$\frac{d^2\varphi}{dx^2} - \frac{1}{L^2}\varphi(x) = 0, \quad x > 0$$

Boundary conditions

(a) 
$$\lim_{x \to 0^+} -D \frac{d\phi}{dx} = \frac{S_0}{2}$$
  $S_0$ : neutron source[s<sup>-1</sup> · cm<sup>-2</sup>]  
(b)  $\lim_{x \to \infty} \phi(x) < \infty$ 

The general solution

$$\varphi(x) = \operatorname{Aexp}\left(-\frac{x}{L}\right) + \operatorname{Bexp}\left(\frac{x}{L}\right)$$

From boundary condition (b),

$$B=0$$

From boundary condition (a),

$$\lim_{x \to 0^+} -D\left(-\frac{A}{L}\exp\left(-\frac{x}{L}\right)\right) = \frac{AD}{L} = \frac{S_0}{2}$$
$$\therefore A = \frac{S_0L}{2D}$$

Hence 
$$\phi(x) = \frac{S_0 L}{2D} \exp\left(-\frac{x}{L}\right), \quad x > 0$$

Copyright © 2016 Toru Obara All Rights Reserved.

By symmetry

$$\varphi(x) \!=\! \frac{S_0 L}{2 D} \exp\left(\!\frac{x}{L}\!\right), \quad x < 0$$