Exchange Economy with Indivisible Goods – the Shapley-Scarf "Housing" Market

I. Overview

- Previous models – assumed transferable utility, money
- Assignment games – players were either buyers or sellers
- This time – each player initially owns an indivisible good
- Here, consider market games with no medium of exchange (no money) – exchange through barters
- Some similarities with the matching models described in the previous lectures.

II. The Model of Shapley and Scarf (1974) and the Core

- Each $i \in N$ owns one unit of an indivisible good that is differentiated – that is, i's good is different from j's good.
- Each $i \in N$ wants at most one unit of the indivisible good.
- Applications of this model
  - Assignment of students to on-campus houses (Abdulkadiroglu and Sönmez (1999))
  - Kidney exchange – exchange of donors (Roth, Sönmez, and Ünver (2004))
- Model defined by two components: $(N, (\precsim_i)_{i \in N})$ where N is as before and $\precsim_i$ represents the preferences of $i \in N$ defined over N where

$$j \precsim_i k \Leftrightarrow \ i \text{ weakly prefers } k\text{'s good over } j\text{'s good.}$$

- Strict preference is denoted by $\prec_i$, and indifference by $\sim_i$.
- An **allocation** is a permutation of N. That is, it is a function $x : N \to N$ such that x is a bijection – that is, if $i \neq j$, then $x(i) \neq x(j)$.
- An allocation x is **dominated by** another allocation y via $S \subseteq N$ if
  - $\{y(i) : i \in S\} = S$ and
  - $x(i) \prec_i y(i)$ $\forall i \in S$.

- An allocation x is **weakly dominated by** another allocation y via $S \subseteq N$ if
  - $\{y(i) : i \in S\} = S$ and
  - $x(i) \precsim_i y(i)$ $\forall i \in S$ with $x(j) \prec_j y(j)$ for some $j \in S$.
- Based on these two domination relations, two types of cores can be defined.

Definition. The **core**, denoted by C is the set of allocations that are not dominated by any allocation. The **strong core or strict core**, denoted by SC is the set of allocations that are not weakly dominated by any allocation.

III. Nonemptiness of the Core and the Top Trading Cycles Method

- The core is always nonempty. An allocation in the core can be found using the top trading cycles method, which is introduced in Shapley and Scarf (1974) and attributed to David Gale.
- For a subset $T \subseteq N$, define

$$B(T, \precsim_i) = \{k \in T | j \precsim_i k,\ \forall j \in T\}.$$

  This set represents the set of goods (more precisely, the owner of the goods) that i likes the most.
- Consider the following procedure called the **top trading cycle method**, which yields as output an allocation with nice properties.

Top Trading Cycles Method. Let $t := 0$ and $N^0 := N$.
Step t: While $N^t \neq \emptyset$, find a set of players $S = \{i_1, i_2, \cdots, i_k\}$ where for each $l = 1, 2, \cdots, k$, $i_{l+1} \in B(N^t, \precsim_{i_l})$ and $i_{k+1} = i_1$. Such a set exists since N^t is finite. Assign to i_l, the good owned by i_{l+1}, where the same convention $i_{k+1} = i_1$ applies. Let $N^{t+1} := N^t \setminus S$ and go to step $t + 1$. (Stop if $N^{t+1} = \emptyset$.)

- Let $\pi : N \to \mathcal{R}$ denote the price function where $\pi(i)$ represents the price of the good owned initially by $i \in N$.
- An allocation-price pair (x, π) is a **competitive equilibrium** if it satisfies the following two conditions for all $i \in N$:

1. $\pi(x(i)) \leq \pi(i)$.
2. $x(i) \succsim_i j$ for all j such that $\pi(j) \leq \pi(i)$.

That is, $x(i)$ is the good that i likes among those whose prices are less than or equal to $\pi(i)$, the worth of i's endowment.

- Note that if $\{i_1, i_2 \cdots, i_K\}$ is a trading cycle corresponding to a competitive equilibrium (x, π) – that is, $i_k = x(i_{k-1})$ where $i_K = i_0$ – then,

$$\pi(i_1) = \pi(i_2) = \cdots = \pi(i_K).$$

- An allocation x is a **competitive allocation** if there exists π such that (x, π) is a competitive equilibrium. Let CA be the set of competitive allocations.

Theorem 1. (Shapley and Scarf (1974)) The set of allocations that result from the top trading cycles coincides with the set CA. Moreover,

$$\emptyset \neq CA \subseteq C.$$

IV. When Preferences are Strict – A Case for the Strong Core

- The core is always nonempty, but it may contain some undesirable allocations. The example below from Roth and Postlewaite (1977):

$$3 \succ_1 2 \succ_1 1$$
$$1 \succ_2 2 \succ_2 3$$
$$2 \succ_3 3 \succ_3 1$$

The top trading cycles method yields the allocation $(3, 1, 2)$ (that is, the good initially owned by 3 is given to 1, etc.). However, there is another allocation $(2, 1, 3)$ that is also in the core.

- However, after the allocation $(2, 1, 3)$ is realized, players 1 and 3 prefer to trade good 2 and good 3 (those goods initially owned by players 2 and 3 respectively).
- Despite the fact that $(2, 1, 3)$ is in the core, it does not persist due to the possibility of further trading by the players. Roth and Postlewaite (1977) argue that such an allocation is not "stable."

- Roth and Postlewaite (1977) shows that a competitive allocation is stable. Moreover, under strict preferences, CA and SC coincides (and thus $SC \neq \emptyset$.)

- Assume that preferences are strict. That is, preferences $\precsim_i$ are said to be **strict** if

$$j \sim_i k \Rightarrow j = k.$$

> **Theorem 2. (Shapley and Scarf (1974), Roth and Postlewaite (1977))** When preferences are strict, the top trading cycle yields a unique allocation x^*. This x^* also has the following properties.
>
> 1. x^* weakly dominates all other allocations $x \neq x^*$.
>
> 2. $SC = CA = \{x^*\}$.

V. When Preferences are not Strict

- When prerferences are not strict: it is possible for $j \sim_i k$ to hold with $j \neq k$.

- In the top trading cycles method, it is not specified which "best" good an agent should be received when there is a tie. However, in whatever way a tie is broken, the resulting allocation is still a competitive allocation. Therefore, Theorem 1 still holds.

- The strong core can be empty, and Theorem 2 no longer holds. In particular, there may be a housing market in which every competitive allocation is not Pareto efficient.[1] An example, attributed to Jun Wako, is given in Sotomayor (2005) reproduced below.

$$2 \succ_1 3 \succ_1 1$$
$$1 \sim_2 3 \succ_2 2$$
$$2 \succ_3 1 \succ_3 3$$

- Strict preferences are sufficient for SC to be nonempty but are not necessary (consider the case in which all players are indifferent between any two goods). A nec-

[1] An equivalent definition of Pareto efficiency is that x is **Pareto efficient** iff x is not weakly dominated by an allocation y via N.

essary and sufficient condition for the nonemptiness of the strong core is given in Quint and Wako (2004).

- Every strong core allocation is contained in the set of competitive allocations.

Theorem 3. (Wako (1984))
$$SC \subseteq CA$$

- When preferences are strict, SC is a singleton allocation which weakly dominated all other allocations. When preferences are not strict, then the strong core SC may contain multiple allocations, but each agent is indifferent between any two allocations in SC. Moreover, every allocation in the strong core weakly dominates all other allocations.

Theorem 4. (Wako (1991))

1. The following holds for the strong core SC.
$$x, y \in SC \Rightarrow x(i) \sim_i y(i)\ \forall i \in N.$$
2. Let $x \notin SC$. Then for every $y \in SC$, x is weakly dominated by y.

- Theorem 4 establishes the analogue of Theorem 2 for general preferences. However, it says nothing for markets in which $SC = \emptyset$.
- Recall that Theorem 2 involved also CA and that $CA \neq \emptyset$ always holds. The analogue for the part involving CA is the following.

Theorem 5. (Wako (1991)) Suppose that x is weakly dominated by some allocation. Then, there exists some $y \in CA$ such that x is weakly dominated by y.

References

Abdulkadiroglu, A. and T. Sönmez (1999). House allocation with existing tenants. *Journal of Economic Theory 88*, 233–260.

Quint, T. and J. Wako (2004). On houseswapping, the strict core, segmentation, and linear programming. *Mathematics of Operations Research 29*, 861–877.

Roth, A. E. and A. Postlewaite (1977). Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics 4*, 131–137.

Roth, A. E., T. Sönmez, and M. U. Ünver (2004). Kidney exchange. *Quarterly Journal of Economics 119-2*, 457–488.

Shapley, L. S. and H. Scarf (1974). On cores and indivisibility. *Journal of Mathematical Economics 1*, 23–38.

Sotomayor, M. (2005). An elementary non-constructive proof of the non-emptiness of the core of the housing market of Shapley and Scarf. *Mathematical Social Sciences 50*, 298–303.

Wako, J. (1984). A note on the strong core of a market with indivisible goods. *Journal of Mathematical Economics 13*, 189–194.

Wako, J. (1991). Some properties of weak domination in an exchange market with indivisible goods. *The Economic Studies Quarterly 42*, 189–194.