Matching

I. Overview

- Players divided into two distinct groups. This also was the case for assignment games discussed before with buyers and sellers.
- Objective is to form pairs consisting of one player from each group. $\rightarrow$ stable matchings ($\approx$ core)
- Unlike assignment games: no side-payments are allowed – remove the transferable utility (TU) assumption
- Goal: to find stable matchings and their properties
- Original model from Gale and Shapley (1962).

II. One-to-one Matching Market: Setup and Definitions

- $N = M \cup W$ the set of agents where $M \cap W = \emptyset$.
- $M = \{m_1, m_2, \cdots, m_p\}$ is called the set of men and $W = \{w_1, w_2, \cdots, w_q\}$ the set of women.
- Each $m \in M$ has **strict** preferences over $W \cup \{m\}$, denoted by $\succ_m$.
  - Strict: For $j, k \in W \cup \{m\}$, either $j \succ_m k$ or $k \succ_m j$.
  - $w \succ_m m$: m prefers w over being single (denoted by the outcome that m is matched to himself). Such w is said to be **acceptable** to m.
- Each $w \in W$ also has strict preferences over $M \cup \{w\}$, denoted by $\succ_w$. An acceptable m for w can be similarly defined.
- $(M, W, (\succ_i)_{i \in M \cup W})$ define a (two-sided) **matching market**.

Definition. A **matching** is a one-to-one function $\mu : M \cup W \rightarrow M \cup W$ such that

- $\mu(m) \in W \cup \{m\}$ for all $m \in M$ and $\mu(w) \in M \cup \{w\}$ for all $w \in W$,
- $\mu(\mu(i)) = i$ for all $i \in M \cup W$.

- Let $\mathcal{M}$ denote the set of all matchings.

- Define the relation $\succeq_i$ to be such that $j \succeq_i k$ if $j \succ_i k$ or $j = k$.
- A matching μ is said to be **individually rational** if $\mu(i) \succeq_i i$ for all $i \in M \cup W$. That is, a matching assigns to each i that is acceptable or i is single.
- A pair (m, w) forms a **blocking pair** of matching μ or **blocks** a matching μ if

$$m \succ_w \mu(w)$$

and

$$w \succ_m \mu(m).$$

Definition. A matching μ is **stable** if it is individually rational and there does not exist a pair $(m, w) \in M \times W$ that blocks it.

- A stable matching always exists in a matching market. Below is an algorithm, given by Gale and Shapley (1962), to find a stable matching.

M-proposing DA Algorithm (One-to-one Case)

(Step 1.a) Each man $m \in M$ proposes to $w \in W$ whom he likes the most among those acceptable to m. If no such $w \in W$ exists, m is matched to himself.

(Step 1.b) Each woman $w \in W$ chooses the most preferred $m \in M$ who proposed to w and is acceptable and rejects all other men who have proposed to her. In such a case, m and w are tentatively matched to each other.

$$\cdots$$

(Step k.a) Each man $m \in M$ proposes to $w \in W$ whom he likes the most among those acceptable to m and who has not rejected m at an earlier step. If no such $w \in W$ exists, m is matched to himself.

(Step k.b) Each woman $w \in W$ chooses the most preferred $m \in M$ who proposed to w and her tentative partner and rejects all other men. In such a case, m and w are tentatively matched to each other.

- A version of the DA algorithm where women propose and men choose whether to accept or not – W-proposing algorithm

- The DA algorithm (regardless of who proposes) yields a stable matching.
  - Let μ be the matching obtained in the above algorithm. If μ were not stable, there would exist m and w such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.
  - $w \succ_m \mu(m)$ implies m must have proposed to w at some step.
  - However, m is rejected by w (otherwise, they would be matched together in μ) at some point, where then w chooses some m' with $m' \succ_w m$.
  - w's final partner cannot be someone preferred less to m' so that $\mu(w) \succeq_w m' \succ_w m$, which contradicts $m \succ_w \mu(w)$. ($\succeq_w$: $\succ_w$ or $=$)
- The M-proposing DA algorithm yields the M-optimal matching, labeled by μ^M. That is, if μ is any stable matching, then

$$\mu^M(m) \succeq_m \mu(m)\ \forall m \in M.$$

  - A woman $w \in W$ will be called **achievable to** m if there exists a stable matching that matches w to m. Therefore, it is sufficient to show that in the DA algorithm, any man $m \in M$ will not be rejected by w who is achievable to m.
  - Let k be the first step in which a man $m \in M$ is rejected by some $w \in W$ who is achievable to m.
  - For w to reject m, m must be unacceptable ($w \succ_w m$) or w is tentatively assigned to some $m' \in M$ with $m' \succ_w m$.
  - Let μ be a stable matching such that $\mu(m) = w$. Goal is to show (w, m') forms a blocking pair, which is a contradiction to μ being stable.
  - Who is $\mu(m')$? – Let $w' = \mu(m')$. This w' must be achievable to m', and it **cannot** be the case that $w' \succ_{m'} w$. Otherwise, m' must have proposed to w' before proposing to w but was rejected by w' at a step earlier than k. This contradicts how k was defined.
  - Therefore, $w \succ_m w' = \mu(m')$ (since $w \neq w'$). Recall that $m' \succ_w m = \mu(w)$. Therefore, (m', w) forms a blocking pair.
- A stable matching is also immune to coalitional deviations so that it is a "core" matching – similar result to the simplification of the conditions in the core of the assignment game. (Homework)

III. Mathematical Properties of Stable Matchings

Theorem 1. (Lattice Theorem) Let μ and μ' be two stable matchings. Define λ and ν by

$$\lambda(m) = \max_{\succ_m}\{\mu(m), \mu'(m)\}, \lambda(w) = \min_{\succ_w}\{\mu(w), \mu'(w)\}$$

$$\nu(m) = \min_{\succ_m}\{\mu(m), \mu'(m)\}, \nu(w) = \max_{\succ_w}\{\mu(w), \mu'(w)\}$$

where the max operation is defined in the following way:

$$\max_{\succ_m}\{X, Y\} = \begin{cases} X & \text{if } X \succeq_m Y \\ Y & \text{if } Y \succ_m X \end{cases}$$

and min and operations with respect $\succ_w$ are defined similarly. Then λ and ν are both stable matchings. (It is not trivial that these functions λ and ν need be matchings.)

- Because the set of matchings is finite, an immediate corollary of the above result is the following.

Corollary. (Coincidence of Interest) There exists an M-optimal and W-optimal matching.

- The next result shows that agents in M and W are opposites regarding to their preferences of one stable matching to another.

Theorem 2. (Conflict of Interest) Let μ and μ' be two stable matchings such that $\mu(m) \preceq_m \mu'(m)\ \forall m \in M$. Then, $\mu(w) \succeq_w \mu'(w)\ \forall w \in W$.

- Finally, the last property, called the **Rural Hospital Theorem**, states that those who are not matched in one stable matching is not matched in any other stable matching.
- The name comes from the matching between interns and hospitals – rural hospitals are unpopular, and theorem shows that as long as matchings are stable, rural hospitals still are unpopular.

Theorem 3. (Rural Hospital Theorem) Let μ be a stable matching and $i \in M \cup W$ such that $\mu(i) = i$. Then, for any stable matching μ', $\mu'(i) = i$.

IV. A Related Problem: Roommate Problem

- In the previous model – restriction of matching between men and women. Two men or two women could not form pairs.
- Roommate problem – no such restriction – any two players can form a pair.
- N: the set of players
- Each $i \in N$ has preferences (denoted by $\succ_i$) over N.
- A matching μ in the roommate problem is a one-to-one function from N to itself that satisfies $\mu(\mu(i)) = i$ for each $i \in N$.
- Individual rationality is defined in the same way as in the matching problem.
- A pair $(i, j) \in N \times N$ is said to block a matching μ if the following hold:

$$j \succ_i \mu(i) \text{ and } i \succ_j \mu(j).$$

- A matching μ is said to be **stable** if it is individually rational and not blocked by any pair (i, j).
- Unlike the two-sided matching problem, a stable matching (defined later) may **not** exist. (See Example below)

Example: $N = \{1, 2, 3, 4\}$ and preferences are given by the following.

$$1 : 2 \succ_1 3 \succ_1 4$$
$$2 : 3 \succ_2 1 \succ_2 4$$
$$3 : 1 \succ_3 2 \succ_3 4$$
$$4 : 1 \succ_4 2 \succ_4 3$$

There is no stable matching for this problem.

- Matching problems are special cases of roommate problems. Therefore, roommate problems are broader.

- The following results, which are known to hold for matching markets, also hold for roommate problems.

Theorem 2'. (Conflict of Interest, Roommate Version)
Let μ and μ' be two stable matchings in the roommate problem. Let $i \in N$ be such that $\mu(i) \succ_i \mu'(i)$. Then, for $j = \mu(i)$, $\mu'(j) \succ_j \mu(j)$.

Theorem 3'. (Lone Wolf Theorem) Let μ be a stable matching and $i \in N$ be such that $\mu(i) = i$. Then, for any stable matching μ', $\mu'(i) = i$.

- Proof of the above two theorems given using the techniques of Klaus and Klijn (2010).

References

Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *American Mathematics Monthly 69*, 9–15.

Klaus, B. and F. Klijn (2010). Smith and Rawls share a room: stability and medians. *Social Choice and Welfare 35*, 647–667.