Assignment Games

I. Setup (Shapley and Shubik (1972))

- Two groups of players: sellers and buyers
  - Sellers – own a unit of indivisible good (e.g. house). Q: set of sellers.
  - Buyers – own money, to buy one unit of a good from one of the sellers. Each buyer has no use for more than one unit of the good. P: set of buyers.
  - $P \cap Q = \emptyset$, $P \cup Q = N$: the set of all players.
- The concept of **matching**: each buyer is matched with at most one seller and vice versa. Formal definition is below.

Definition: A **matching** is a one-to-one function $\mu : P \cup Q \to P \cup Q$ such that

- $\mu(i) \in Q \cup \{i\}$ for all $i \in P$ and $\mu(j) \in P \cup \{j\}$ for all $j \in Q$,
- $\mu\left(\mu(i)\right) = i$ for all $i \in P \cup Q$.

- A matching can be described by vector notation. Let $x = (x_{ij})_{(i,j)\in P\times Q}$ where

$$x_{ij} = \begin{cases} 1 & \text{if } \mu(i) = j \\ 0 & \text{otherwise} \end{cases}$$

- a_{ij}: the joint surplus when buyer i and seller j trade with each other. Let $A = (a_{ij})_{(i,j)\in P\times Q}$ be the matrix where entry (i,j) is a_{ij}.
- If a buyer or a seller is unmatched, then that agent's surplus is 0. Therefore, when defining a TU game with characteristic function v, then $v(\{i\}) = 0$ for all $i \in P \cup Q$.
- For all other coalitions: let $S \subseteq P$ and $S' \subseteq Q$, $v(S \cup S')$ is defined as the maximum value of the objective function of the following maximization problem.

$$\max \sum_{(i,j)\in S\times S'} x_{ij} a_{ij} \tag{1}$$

subject to the following constraints:

$$\sum_{j \in S'} x_{ij} \leq 1 \ \forall i \in S$$
$$\sum_{i \in S} x_{ij} \leq 1 \ \forall j \in S'$$
$$x_{ij} \in \{0,1\} \ \forall i \in S, \forall j \in S'$$

- The concept of imputation carries over to the assignment game as well. Let X be the set of imputations.
- Notation: Let $(u,v) \in \mathcal{R}_+^P \times \mathcal{R}_+^Q$ where $u = (u_i)_{i \in P}$ is a vector of payoffs for the buyers and $v = (v_j)_{j \in Q}$ is a vector for the sellers.
- (u,v) is said to be **compatible** with a matching x if

$$\sum_{i \in P} u_i + \sum_{j \in Q} v_j = \sum_{i,j} a_{ij} x_{ij}$$

- The **core** of the assignment game is defined in the usual way using coalitional rationality. That is,

$$C = \left\{ (u,v) \in X \middle| \sum_{i \in S} u_i + \sum_{j \in S'} v_j \geq v(S \cup S') \ \forall S \subseteq P, S' \subseteq Q \right\}$$

- Although all subsets of sellers and buyers have to be considered in the definition of the core, it can be shown that it is sufficient to consider only buyer-seller pairs. That is,
$$C = \{(u,v) \in X | u_i + v_j \geq a_{ij} \ \forall i \in P, j \in Q\}.$$

II. Nonemptiness and Structure of the Core

- Consider the maximization problem for $S = P$, $S' = Q$ and relax the last constraint to form an LP:
$$\max \sum_{(i,j) \in P \times Q} x_{ij} a_{ij} \tag{2}$$

subject to the following constraints:

$$\sum_{j \in Q} x_{ij} \leq 1 \ \forall i \in P$$
$$\sum_{i \in P} x_{ij} \leq 1 \ \forall j \in Q$$
$$x_{ij} \geq 0 \ \forall i \in P, \forall j \in Q$$

- It is known that this relaxed problem has an integral solution. That is x_{ij} are all integers, and by the constraint, should be either 0 or 1. This solution must also be the solution to the original problem. Therefore, the max value of the relaxed LP problem (2) is equal to $v(P \cup Q)$.

- Consider the dual of (2), given below.

$$\min \sum_{i \in P} u_i + \sum_{j \in Q} v_j \tag{3}$$

subject to

$$u_i + v_j \geq a_{ij} \ \forall i \in P, j \in Q$$
$$u_i \geq 0 \ \forall i \in P$$
$$v_j \geq 0 \ \forall j \in Q$$

- From duality theorem, the minimum value of the dual (3) = maximum value of the primal (2). Therefore, there exists $(u, v) \in \mathcal{R}_+^P \times \mathcal{R}_+^Q$ that satisfies the constraints of the dual (3). Therefore, (u, v) is in the core.

- Another implication of the dual relationship: If (u, v) is in the core, then it is compatible with <u>any</u> optimal matching x^*. Therefore, (u, v) can be defined separately from the optimal matching. This property does not hold for general models (see Demange and Gale (1985)).

- Furthermore, it can be shown that the core is a lattice in the space $\mathcal{R}_+^P \times \mathcal{R}_+^Q$ in the following sense.

- Let (u, v) and (u', v') be two allocations in the core. Define the following.

$$\bar{u}_i = \max\{u_i, u'_i\}$$
$$\underline{u}_i = \min\{u_i, u'_i\}$$
$$\bar{v}_j = \max\{v_j, v'_j\}$$
$$\underline{v}_j = \min\{v_j, v'_j\}$$

  Then, $(\bar{u}, \underline{v})$ and $(\underline{u}, \bar{v})$ are in the core where $\bar{u} = (\bar{u}_i)_{i \in P}$ and $\underline{u} = (\underline{u}_i)_{i \in P}$ and $\bar{v}$ and $\underline{v}$ are defined similarly. The proof of the group rationality of (u, v) is given below:

  - Let μ be an optimal matching. Note (u, v) and (u', v') are both compatible to μ.
  - Note $\bar{u}_i = \max\{u_i, u'_i\} = \max\{a_{i\mu(i)} - v_{\mu(i)}, a_{i\mu(i)} - v'_{\mu(i)}\} = a_{i\mu(i)} - \min\{v_{\mu(i)}, v'_{\mu(i)}\}$. Equivalently,
  $$\bar{u}_i = a_{i\mu(i)} - \underline{v}_{\mu(i)} \; \forall i \in P, \mu(i) \in Q.$$
  - For all $i \in P$, $j \in Q$ unmatched, $u_i = u'_i = v_j = v'_j = 0$. (by coalitional rationality). Thus,
  $$\bar{u}_i = \underline{u}_i = \bar{v}_j = \underline{v}_j = 0$$
  - Now, calculate $\sum_{i \in P} \bar{u}_i + \sum_{j \in Q} \underline{v}_j = \sum_{i:i \text{ matched}} (\bar{u}_i + \underline{v}_{\mu(i)})$.

III. Multi-item Auction (Demange, Gale, and Sotomayor (1986)): Setting

- As before P is the set of buyers. In this setting, Q will be called the set of objects.
- Also, for notational ease let O denote the null object in Q so that when a buyer is assigned to O, that buyer is technically unmatched. The null object can be assigned to an arbitrary number of buyers, while any other object can only be assigned to at most one buyer.
- Each seller of an object j is willing to sell it at a price $p_j \geq r_j$. r_j is typically called the reservation price.
- For each object j, each buyer i has a valuation h_{ij} over object j where h_{ij} denotes the greatest i is willing to pay for the object j. Suppose that each h_{ij} is an integer.
- Let $p = (p_j)_{j \in Q}$ denote a vector of prices. Then, the **demand set** of a buyer i, denoted by $D_i(p)$ is given by the following.

$$D_i(p) = \{j \in Q | h_{ij} - p_j \geq h_{ij'} - p_{j'} \; \forall j' \in Q\}$$

That is, the set of objects that maximizes i's surplus.

- A vector of prices p is said to be **competitive** if there exists a matching μ such that for all $i \in P$,

$$\mu(i) \in D_i(p),$$

  and for all objects j that are unmatched, $p_j = r_j$. Collectively, the vector of prices together with the associated matching assignment μ constitute a **competitive equilibrium**.

IV. Graphical Representation of the Problem

- Let $P \cup Q$ be the set of nodes.
- Suppose that the vector of prices p is given. Draw an edge between $i \in P$ and $j \in Q$ if $j \in D_i(p)$.
- Such a graph is **bipartite**. That is, there are no edges between any $i, i' \in P$ or $j, j' \in Q$. All edges are characterized by $(i, j) \in P \times Q$.
- A **complete matching of a graph** G is a subgraph with the same set of vertices of G such that each vertex i is connected to exactly one other vertex j through an edge of the original graph G. In terms of the framework here, each $i \in P$, there exists exactly one $j \in Q$ such that (i, j) is an edge in the original graph.
- Looking for a competitive equilibrium = Existence of a complete matching.
- Hall's theorem: necessary and sufficient condition for the existence of a complete matching.

Hall's Theorem. Consider the bipartite graph described above. A complete matching exists if and only if for every subset $S \subset P$,

$$\left| \bigcup_{i \in S} D_i \right| \geq |S|.$$

Given a set of objects, the number of objects in that set is greater than or equal to the number of bidders demanding goods only in that set.

V. Description of the Auction

1. Each step is indexed by $t = 0, 1, \cdots$, and let $p_j(0)$ denote the price of j at step t and $p(0) = (p_j)_{j \in Q}$. Initially, set $p_j(0) = r_j$, the reservation price of each object.

2. Calculate the demand set $D_i(p(0))$ and find an overdemanded set $S \subset P$ such that

$$\left| \bigcup_{i \in S} D_i(p(0)) \right| < |S| \tag{4}$$

   If none exists, then Hall's theorem implies that there exists a complete matching and hence a competitive equilibrium. If such exists, take a minimal such T that satisfies (4). That is, there does not exist $T' \subsetneq T$ that also satisfies (4).

3. Raise the price of each object in $\bigcup_{i \in T} D_i$ by 1, while keeping the price of the others constant and label this price vector as $p(1)$. That is,

$$p_j(1) = \begin{cases} p_j(0) + 1 & \text{if } j \in D_i(p(0)) \text{ for some } i \in T \\ p_j(0) & \text{otherwise} \end{cases}$$

4. Repeat the process until there is no overdemanded set.

5. This auction yields a price vector p that it is the minimum competitive price. Also, associated with this vector p, construct u_i and v_j such that

$$u_i = h_{ij} - p_j, \ v_j = p_j - r_j.$$

   The vector $((u_i)_{i \in P}, (v_j)_{j \in Q})$ is the buyer-optimal core allocation.

References

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