Basic Theory of Transferable Utility (TU) Games: Nucleolus

I. Overview

- In the previous lectures:
  - TU game was defined.
  - Several concepts were defined: imputation, core.
- Core, relatively simple concept, had two weaknesses or drawbacks:
  - It could be empty.
  - It could be too large.
- Two relatively popular solution concepts that are singleton:
  - Nucleolus
  - Shapley value

II. Definition of Nucleolus

- Let (N, v) be a game and $x \in \mathcal{R}^n$ and $S \subseteq N$. The **excess** of coalition S at x, denoted by $e(S, x)$, is defined by

$$e(S, x) = v(S) - \sum_{i \in S} x_i \tag{1}$$

  where by convention $e(\emptyset, x) = 0$ for all x.

- Using this notation, the core is equivalent to the following.

$$\mathcal{C}(N, v) = \{x \in X(N, v) | e(S, x) \leq 0 \ \forall S \subseteq N\} \tag{2}$$

- Let $x \in \mathcal{R}^n$. Define $\theta(x)$ to be the vector of excesses associated with x in non-increasing order. Because $e(\emptyset, x) = 0$ and $e(N, x) = 0$ for any imputation x, these two coalitions are excluded in this vector θ. That is,

$$\theta(x) = (\theta_1(x), \theta_2(x), \cdots, \theta_{2^n-2}(x)) \in \mathcal{R}^{2^n-2}$$

  where

$$\theta_1(x) \geq \theta_2(x) \geq \cdots \theta_{2^n-2}(x)$$

  and each $\theta_i(x)$ is associated with the excess of some coalition.

- Define the **lexicographic ordering** $\leq_{lex}$ on these vectors in the following way. For two vectors a and b, $a \leq_{lex} b$ if either $a = b$ or there exists a number k such that
  - $a_l = b_l \; \forall l \in \{1, 2, \cdots, k-1\}$ and
  - $a_k < b_k$.
- The ordering $\leq_{lex}$ is a partial ordering, and for every $a, b \in \mathcal{R}^n$, either $a \leq_{lex} b$ or $b \leq_{lex} a$.

Definition. The **nucleolus** of a game (N, v), denoted by $\mathcal{N}(N, v)$ is the set of imputations x for which $\theta(x)$ is lexicographically minimum among all imputations. Formally,

$$\mathcal{N}(N, v) = \{x \in X(N, v) | \theta(x) \leq_{lex} \theta(y) \; \forall y \in X(N, v)\} \tag{3}$$

- One property of the nucleolus is that it is always a subset of the core if the core is nonempty.

Proposition. Let (N, v) be a TU game such that the $\mathcal{C}(N, v) \neq \emptyset$. Then,

$$\mathcal{N}(N, v) \subseteq \mathcal{C}(N, v).$$

- Sketch of proof: Suppose that there exists an imputation $x \in \mathcal{N}(N, v)$ such that $x \notin \mathcal{C}(N, v)$, which implies that for some coalition S, $e(S, x) > 0$. Now, compare $\theta(x)$ to $\theta(y)$ where $y \in \mathcal{C}(N, v)$ to reach a contradiction.

III. Nonemptiness of the Nucleolus

- It can be shown that for any game (N, v), $\mathcal{N}(N, v) \neq \emptyset$, using Weierstrauss' Theorem.
- Note: For any coalition $S \subseteq N$, $e(S, \cdot) = v(S) - \sum_{i \in S} x_i$ is a continuous function of $x = (x_i)_{i \in N}$.
- To show that $\theta_k(\cdot)$ is a continuous function of x for each $1 \leq k \leq 2^n - 2$, the following result is useful.

Proposition. For each $1 \leq k \leq 2^n - 2$,

$$\theta_k(x) = \max_{\mathbf{T} \subseteq 2^N \setminus \{\emptyset, N\}, |\mathbf{T}| = k} \min_{S \in \mathbf{T}} e(S, x) \tag{4}$$

- The interpretation of the right hand side of (4):
  - The "min" operations picks the smallest (or k-th highest) excess of the k coalitions in $\mathbf{T}$ with respect to x.
  - In order to pick the k-th highest among all options – the "max" operation
- Because $\theta_k(\cdot)$ is defined by a finite number of max and min of continuous functions, θ_k is a continuous function.
- To establish existence, consider the following series of optimization problems.

Problem 1. Find $x \in X(N, v)$ that solves the following:

$$\min_{x \in X(N,v)} \theta_1(x) \tag{5}$$

Let X_1 denote the set of imputations that solves Problem 1 (or (5)).

Problem k. Find $x \in X_{k-1}$ that solves the following:

$$\min_{x \in X_{k-1}} \theta_k(x) \tag{6}$$

Let X_k denote the set of imputations that solves Problem k (or (6)).

- By Weierstrauss's theorem, $X_1 \neq \emptyset$ and compact since $X(N, v) \neq \emptyset$ is compact and θ is continuous.
- Continuing in this manner, $X_k \neq \emptyset$ is compact for all $k = 1, 2, \cdots, 2^n - 2$. In particular, $\emptyset \neq X_{2^n-2} = \mathcal{N}(N, v)$.

IV. The Nucleolus is a Singleton

- In Section III, it was established that $\mathcal{N}(N,v) \neq \emptyset$.
- In this section, it is shown that for every game (N,v), $\mathcal{N}(N,v)$ consists of only one imputation.

Theorem. Let (N,v) be any TU game. If $x,y \in \mathcal{N}(N,v)$, then $x = y$.

Below is a sketch of the proof of this statement. Suppose throughout that $x,y \in \mathcal{N}$ and $x \neq y$. Let $z = (x+y)/2$. The objective is to show that $\theta(z) <_{lex} \theta(x)$.

1. $x,y \in \mathcal{N} \Rightarrow \theta(x) = \theta(y)$. That is, $\theta_l(x) = \theta_l(y)$ for all $l = 1, 2, \cdots, 2^n - 2$.
2. Let $S_1, S_2, \cdots, S_{2^n-2}$ be the coalitions that give the excess values in $\theta(x)$. That is, for each l
$$e(S_l, x) = \theta_l(x)$$
3. Similarly define $T_1, T_2, \cdots, T_{2^n-2}$ for the excess values in $\theta(y)$. That is, for each l,
$$e(T_l, y) = \theta_l(y)$$
4. From how the coalitions S_l and T_l were defined, there may be many ways to order the S_l's and T_l's if, for example, consecutive entries in $\theta(x)$ (and in $\theta(y)$ since $\theta(x) = \theta(y)$) are equal. Therefore, reorder the S_l's and T_l's such that the number k that satisfies the condition below is maximized.
    - $S_l = T_l$ for all $l \leq k-1$
    - $S_k \neq T_k$

   Such a k must exist since $x \neq y$, so that there exists $i \in N$ such that $e(\{i\}, x) \neq e(\{i\}, y)$.
5. Note the following property of the excess $e(S, \cdot)$ as a function of the imputation.

Lemma. Let $S \subseteq N$. Then, for any $x, y \in X$ and $0 \leq \lambda \leq 1$,
$$e\left(S, ((1-\lambda)x + \lambda y)\right) = (1-\lambda)e(S,x) + \lambda e(S,y)$$

6. From the lemma, $e(S_l, z) = \frac{1}{2}e(S_l, x) + \frac{1}{2}e(S_l, y) = e(S_l, x) = e(T_l, y)$ for $l \leq k-1$,

7. Now, consider the set of coalitions $\mathcal{S} = \{S \subset N | e(S, x) = e(S_k, x)\}$ and $\mathcal{T} = \{T \subset N | e(T, y) = e(T_k, y)\}$. Note the following facts.

   - $\mathcal{S} \neq \emptyset$ and $\mathcal{T} \neq \emptyset$
   - $\mathcal{S} \cap \mathcal{T} = \emptyset$ – otherwise, if $S \in \mathcal{S} \cap \mathcal{T}$, then coalitions can be re-ordered, contradicting how k was defined.
   - Recalling that $e(S_k, x) = \theta_k(x) = \theta_k(y) = e(T_k, y)$, for each $S \in \mathcal{S}$, $e(S, x) > e(S, y)$ and for $T \in \mathcal{T}$, $e(T, y) > e(T, x)$.
   - By definition of the sets $\mathcal{S}$ and $\mathcal{T}$, $e(S, x) = e(T, y)$ for all $S \in \mathcal{S}$, $T \in \mathcal{T}$.

8. Also, $e(S, z) < e(S, x)$ for all $S \in \mathcal{S}$ and $e(T, z) < e(T, y)$ for all $T \in \mathcal{T}$. Note also that $e(S, z) < e(S, x) \leq e(S_{k-1}, x) = e(S_{k-1}, z)$.

9. Let R^* be a coalition such that $e(R^*, z) = \max_{R \neq S_1, \cdots, S_{k-1}} e(R, z)$. Then, $\theta_k(z) = e(R^*, z)$ and $e(R^*, z) < e(S, x) = \theta_k(x)$ or $e(R^*, z) < e(T, y) = \theta_k(y)$.

10. Thus, $\theta(z) <_{lex} \theta(x) = \theta(y)$, contradicting $x, y \in \mathcal{N}$.

V. Calculation of the Nucleolus – Overall Procedure and Examples

- One way to calculate the nucleolus can be calculated through solving a series of linear programs.

Problem 1'. Find M and $x \in X$ that solves

$$\min M \tag{7}$$

subject to

$$e(S, x) \leq M \ \forall S \subseteq N, S \neq \emptyset, N$$

Let M' denote the optimal value of (7). Let X_1' denote the set of imputations that satisfies the constraints under M'. If $X_1' = \{x\}$ (a singleton), then x is the nucleolus. Let $\mathcal{S}_1 = \{S \in 2^N \setminus \{N, \emptyset\} | e(S, x) = M'\}$ and let $\mathcal{S}_1' = \mathcal{S}_0 \setminus \mathcal{S}_1$ where $\mathcal{S}_0 = 2^N \setminus \{\emptyset, N\}$.

Problem k. Find $x \in X_{k-1}'$ and M_k that solves the following:

$$\min M_k \tag{8}$$

subject to

$$e(S,x) \leq M_k \ \forall S \in \mathcal{S}'_{k-1}$$

Let X'_k denote the set of imputations that solves Problem k (or (8)) and M'_k be the solution to (8).
Continue until X'_k is a singleton.

Example 1:

$$N = \{1,2,3\}$$

$$v(\{1,2,3\}) = 10$$

$$v(\{1,2\}) = 4, v(\{1,3\}) = 3, v(\{2,3\}) = 8$$

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$

Set up the first problem as follows:

$$\min M$$

subject to

$$\begin{aligned} 4 - (x_1 + x_2) &\leq M \\ 3 - (x_1 + x_3) &\leq M \\ 8 - (x_2 + x_3) &\leq M \\ -x_1 &\leq M \\ -x_2 &\leq M \\ -x_3 &\leq M \end{aligned}$$

Because $x \in X$, there is also the condition $x_1 + x_2 + x_3 = 10$. By using this condition, the first three inequalities can be rewritten as follows:

$$\begin{aligned} -6 + x_3 &\leq M \\ -7 + x_2 &\leq M \\ -2 + x_1 &\leq M \end{aligned}$$

Rearraging the inequalities leads to the following.

$$-M \leq x_1 \leq M+2$$
$$-M \leq x_2 \leq M+7$$
$$-M \leq x_3 \leq M+6$$

Also, by the restriction $x_1 + x_2 + x_3 = 10$, the following also needs to be satisfied (from the above inequalities):

$$-3M \leq (x_1 + x_2 + x_3) = 10 \leq 3M + 15$$

The minimum M such that the inequalities hold without contradiction is $M = -1$, which is the excess of the coalitions $\{2,3\}$ and $\{1\}$. This yields $x_1 = 1$ with x_2 and x_3 still undetermined except for $1 \leq x_2 \leq 6$ and $1 \leq x_3 \leq 5$, so the process continues.

Now, substitute $x_1 = 1$ whereever they appear, delete the inequalities corresponding to $\{2,3\}$ and $\{1\}$, and let M' be the next highest excess value. The second problem is as follows:

$$\min M'$$

subject to

$$3 - x_2 \leq M'$$
$$2 - x_3 \leq M'$$
$$-x_2 \leq M'$$
$$-x_3 \leq M'$$

Using the fact that $x_2 + x_3 = 10 - x_1 = 9$, the following set of inequalities, with respect to x_2 can be obtained:

$$-M' + 3 \leq x_2 \leq M' + 7$$

The minimum M' is $M' = -2$, and $x_2 = 5$, implying $x_3 = 9 - x_2 = 4$. Because, the only x that satisfies the inequalities with $M' = -2$ is $(1,5,4)$, the resulting vector $(1,5,4)$ is the nucleolus. □

Example 2:

Consider a TU game with $N = \{1, 2, 3\}$ and v given by

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

The minimization problem to consider is given by

$$\min M$$

subject to

$$\begin{aligned} 1 - (x_1 + x_2) &\leq M \\ 1 - (x_1 + x_3) &\leq M \\ 1 - (x_2 + x_3) &\leq M \\ -x_1 &\leq M \\ -x_2 &\leq M \\ -x_3 &\leq M \end{aligned}$$

Using the same technique as Example 1, we obtain the following

$$\begin{aligned} -M \leq x_1 \leq M \\ -M \leq x_2 \leq M \\ -M \leq x_3 \leq M \end{aligned}$$

Also, by the restriction $x_1 + x_2 + x_3 = 1$, the following also needs to be satisfied (from the above inequalities):

$$-3M \leq 1 \leq 3M \tag{9}$$

The condition $M \geq 1/3$ is the strongest condition. Thus, $M = 1/3$. Note that one of the inequalities in (9) is used. Plugging M back into the inequalities yields

$$\begin{aligned} -1/3 \leq x_1 \leq 1/3 \\ -1/3 \leq x_2 \leq 1/3 \\ -1/3 \leq x_3 \leq 1/3 \end{aligned}$$

The only (x_1, x_2, x_3) that satisfies the above and is also an imputation is $(1/3, 1/3, 1/3)$, which must be the nucleolus.

Example 3 (Calculation using Excess Vectors and Definition of the Nucleolus):
Consider a voting game with players $N = \{1, 2, 3\}$ such player 1 and player 2 are veto players. Formally, the TU game is given by N and the function v defined by

$$v(N) = v(\{1, 2\}) = 1$$

$$v(S) = 0,\ S \neq N, \{1, 2\}.$$

From the previous lecture notes, the core is given by the following set:

$$\mathcal{C}(N, v) = \{(\alpha, 1 - \alpha, 0) \in \mathcal{R}^3 | 0 \leq \alpha \leq 1\}.$$

Thus, the nucleolus must be of the form $(\alpha, 1 - \alpha, 0)$ for $0 \leq \alpha \leq 1$. Consider the case in which $\alpha \geq 1 - \alpha$ and let $y = (\alpha, 1 - \alpha, 0)$. Then, the excess vector of y is

$$\theta(y) = (0, 0, -(1 - \alpha), -(1 - \alpha), -\alpha, -\alpha).$$

and for the case in which $1 - \alpha \geq \alpha$,

$$\theta(y) = (0, 0, -\alpha, -\alpha, -(1 - \alpha), -(1 - \alpha)).$$

This vector is at its lexicographic minimum if $\alpha = 1/2$. Thus, the nucleolus is $(1/2, 1/2, 0)$.

VI. Related Concepts – Least Core and Prenucleolus

- It is known that if $\mathcal{C} \neq \emptyset$, then $\nu \in \mathcal{C}$. However, the core can be empty in some games.
- Let X^* be the set of vectors of $\mathcal{R}^n$ that satisfies group rationality. That is,

  $$X^*(N, v) = \{x \in \mathcal{R}^n | \sum_{i \in N} x_i = v(N)\}$$

  An element $x \in X^*$ is called a **preimputation**, and X^* is called the **set of preimpuatations**.
- For $\varepsilon \in \mathcal{R}$, define the ϵ-core, $\mathcal{C}_\varepsilon$:

  $$\mathcal{C}_\epsilon(N, v) = \{x \in X^*(N, v) | e(S, x) \leq \varepsilon, \forall S \subseteq N\}$$

- When $\varepsilon = 0$, $\mathcal{C}_\varepsilon = \mathcal{C}$. Moreover, for $\varepsilon < \varepsilon'$, then $\mathcal{C}_\varepsilon \subseteq \mathcal{C}_{\varepsilon'}$.
- There exists ε large enough such that $\mathcal{C}_\varepsilon \neq \emptyset$.
- Define the **least core**, $\mathcal{LC}$ of (N, v) by

$$\mathcal{LC} = \bigcap_{\varepsilon, \mathcal{C}_\varepsilon \neq \emptyset} \mathcal{C}_\varepsilon \tag{10}$$

  By definition, $\mathcal{LC} \neq \emptyset$.

- The definition of the least core (10) can be rewritten as the following:

$$\mathcal{LC} = \mathcal{C}_{\varepsilon_0} \tag{11}$$

  where

$$\varepsilon_0 = \min_{x \in X} \max_{S \subset N, S \neq \emptyset} e(S, x)$$

- The **prenucleolus** of (N, v) is the set of $x \in X^*(N, v)$ such that there is no $y \in X^*(N, v)$ such that $\theta(y) <_{lex} \theta(x)$.