Basic Theory of Transferable Utility (TU) Games: Core

I. Overview

- Second model: Cooperative game with side payments (including games with three or more players) $\rightarrow$ transferable utility (TU) coalitional games
- Abstract mathematical model representing a situation in which a subset of players, called a coalition, can form and gain surplus by doing so.
- Questions in cooperative game theory:

  **(Q1)** What coalitions would be formed?

  **(Q2)** How should the surplus be allocated among the players?

II. Definitions

- Given a set $N$, let $2^N$ denote the set of subsets of $N$. That is, $2^N = \{S|S \subseteq N\}$.

**Definition.** A **transferable utility (TU) game in characteristic function form** (or **TU game** or **game**) is given by $(N, v)$ where

- $N = \{1, 2, \cdots, n\}$ is a finite set of players
- $v : 2^N \rightarrow \mathcal{R}$, called the **characteristic function**, is such that $v(S)$ represents the total amount that $S$ can guarantee (also called the worth of $S$). By convention, $v(\emptyset) = 0$.

A subset $S \subseteq N$ is called a **coalition**. $(N, v)$ in some texts is called a **coalitional game** with $v$ as the **coalitional function**.

- A TU game $(N, v)$ is **superadditive** if for every coalition $S$ and $T$ such that $S \cap T = \emptyset$,
$$v(S \cup T) \geq v(S) + v(T)$$
- A TU game is said to be **inessential** if for every $S \subseteq N$,
$$v(S) = \sum_{i \in S} v(\{i\}) \tag{1}$$

  $(N, v)$ is said to be **essential** if it is not inessential. That is, for some $S \subseteq N$ the inequality in (1) is not satisfied.

- A superadditive game $(N, v)$ is essential if and only if $v(N) > \sum_{i \in N} v(\{i\})$.
- Most examples of TU games are superadditive. Loosely speaking, for superadditive games, it is assumed that the coalition $N$ (also called the grand coalition) is formed – answers question (Q1).
- Only need to focus on (Q2) – how to allocate $v(N)$.
- An $n$-vector $x = (x_1, x_2, \cdots, x_n) \in \mathcal{R}^n$ is an **imputation** if it satisfies the following:
  - $\sum_{i \in N} x_i = v(N)$ (group rationality)
  - $x_i \geq v(\{i\})\ \forall i \in N$ (individual rationality)

  Denote the set of imputations of $(N, v)$ by $X(N, v)$ or by $X$ if there is no confusion as to the game being analyzed.
- The two conditions for the imputation – minimal conditions under which an allocation method of $v(N)$ should satisfy.
- Extend "individual rationality" for coalitions $\rightarrow$ coalitional rationality. This then leads to the following concept.

**Definition.** Let $(N, v)$ be a game. The following set below, denoted by $\mathcal{C}(N, v)$, is called the **core** of $(N, v)$.

$$\mathcal{C}(N, v) = \{x \in X(N, v) | \sum_{i \in S} x_i \geq v(S)\ \forall S \subseteq N\}. \tag{2}$$

The condition in the right-hand side of (2) is called **coalitional rationality**.

- An imputation $x$ is said to be **dominated** by imputation $y$ if there exists a coalition $S \subset N$ such that the following hold.
  - $\sum_{i \in S} y_i \leq v(S)$
  - $y_i > x_i\ \forall i \in S$

  If the above holds, it is also said that coalition $S$ **blocks** imputation $x$ via imputation $y$.
- The **dominance core** ($\mathcal{DC}(N, v)$) is defined as the set of imputations that are not dominated. It can be shown that $\mathcal{C}(N, v) \subseteq \mathcal{DC}(N, v)$ for all games $(N, v)$. Moreover, if $(N, v)$ is superadditive, then $\mathcal{C}(N, v) = \mathcal{DC}(N, v)$.

III. Other Definitions

- Let $(N, v)$ and $(N, v')$ be two games. $(N, v')$ is **strategically equivalent** to $(N, v)$ if there exist $\alpha > 0$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathcal{R}^n$ such that

$$v'(S) = \alpha v(S) + \sum_{i \in S} \beta_i \; \forall S \subseteq N$$

- Define the relation $(N, v) \sim (N, v')$ if $(N, v')$ is strategically equivalent to $(N, v)$. It can be shown that $\sim$ is an equivalence relation.

- A game $(N, v')$ is a **zero-normalization** if $(N, v)$ if $(N, v')$ is strategically equivalent to $(N, v)$ and $v'(\{i\}) = 0 \; \forall i \in N$. For every game $(N, v)$ there exists a zero-normalization.

- A game $(N, v)$ is **monotonic** if for all $S, T \in 2^N$ with $S \subseteq T$,

$$v(S) \leq v(T)$$

- For any game $(N, v)$, there is a monotonic game that is strategically equivalent to $(N, v)$.

- A game $(N, v)$ is **zero-monotonic** if for every $S, T \in 2^N$ with $S \subseteq T$,

$$v(T) \geq v(S) + \sum_{i \in T \setminus S} v(\{i\})$$

  It is easily seen that every superadditive game is zero-monotonic.

- Equivalently, a game is zero-monotonic if its zero-normalization is monotonic.

- Let $(N, v')$ be a game that is strategically equivalent to $(N, v)$ with $\alpha > 0$ and vector $\beta \in \mathcal{R}^n$. If $x \in \mathcal{C}(N, v)$, then $x' \in \mathcal{C}(N, v')$ where $x' \in \mathcal{R}^n$ is defined by $x'_i = \alpha x_i + \beta_i$. The converse also holds true. (**covariance property of the core**)

IV. Mathematical Aside: Equivalence Relation

- Example: In expressions such as "$x \leq y$" or "$x = y$," "$\leq$" and "$=$" are called **binary relations**.[1]

[1] Technically, given a set $X$, a binary relation is associated with a subset of $K \subset X \times X$ where $x \leq y \Leftrightarrow (x, y) \in K$.

- A binary relation $\sim$ is called an **equivalence relation** on $X$ if the following hold for all $x, y, z \in X$.

  1. $x \sim x$ (Reflexivity)
  2. $x \sim y \Leftrightarrow y \sim x$ (Symmetry)
  3. $x \sim y$ and $y \sim z \Rightarrow x \sim z$. (Transitivity)

V. Examples

1. Let $N = \{1, 2, 3\}$ be the set of players. Suppose that each player owns a unit of input, and for every two units of input, a unit of output is produced. The profit from selling this unit of output is 1. The game $(N, v)$ associated with this situation is given by the following.

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

   The core of this game is empty. That is, $\mathcal{C}(N, v) = \emptyset$.

2. Consider the same setup as 1. but in order to produce a unit of output, player 1's input is necessary. The game $(N, v)$ associated with this situation is given by

$$v(S) = \begin{cases} 1 & \text{if } 1 \in S \text{ and } |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

   The core $\mathcal{C}(N, v) = \{(1, 0, 0)\}$.

3. Voting Games

   - A game $(N, v)$ is **simple** if $v(N) = 1$ and $v(S) \in \{0, 1\}$ for all $S \subseteq N$.
   - A game $(N, v)$ is a **voting game** if it is simple and monotonic.
   - A voting game $(N, v)$ is **proper** if for all $S \subseteq N$ such that $v(S) = 1$, $v(N \setminus S) = 0$.
   - Given a proper voting game $(N, v)$, the set of veto players is given by $V = \bigcap\{S \subseteq N | v(S) = 1\}$.

   Then, $\mathcal{C}(N, v) \neq \emptyset$ if and only if $V \neq \emptyset$. Moreover, when $V \neq \emptyset$,

$$\mathcal{C}(N, v) = \{x \in X(N, v) | x_i = 0 \ \forall i \in N \setminus V\}.$$

4. Cost Sharing Games

- Let $N$ be the set of towns, each of which needs to draw water from a lake.
- Suppose that for each $S \subseteq N$, the cost of building pipelines to provide water to towns in $S$ costs an amount $c(S) > 0$.
- Costs $c$ can be viewed as a function from $2^N$ to $\mathcal{R}$. Suppose that $c$ is **subadditive**, that is, for all $S, T \in 2^N$ with $S \cap T = \emptyset$,

$$c(S) + c(T) \geq c(S \cup T).$$

- One way to define a TU game $(N, v_1)$ based on this situation is by

$$v_1(S) = -c(S),$$

  but for each nonempty $S$, $v_1(S) < 0$.
- To fix this, the following TU game $(N, v_2)$ is strategically equivalent to $(N, v_1)$ and is such that $v_2(S) \geq 0$, where $v_2$ is defined by

$$v_2(S) = -c(S) + \sum_{i \in S} c(\{i\}).$$

VI. Convex Games – Sufficient Condition for Nonemptiness of the Core

- From the first example, the core of a superadditive game may be empty $\rightarrow$ for a sufficient condition, need a stronger concept.

**Definition.** $(N, v)$ is a **convex** game if for every $S, T \in 2^N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \tag{3}$$

- From (3), it can be seen easily that a convex game is superadditive.
- An equivalent (and sometimes useful) formulation of a convex game is given in the following.

**Proposition.** A game $(N, v)$ is convex $\Leftrightarrow$ for every $S, T \in 2^N$ with $S \subseteq T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad (4)$$

- Sketch of Proof:

  $(\Rightarrow)$ : Let $S, T \in 2^N$ be such that $S \subseteq T$. Let $S' = S \cup \{i\}$ and $T' = T$ and use (3).

  $(\Leftarrow)$ : Let $S, T \in 2^N$ be any pair of coalitions. For notational ease, let $R := S \cap T$ and $T \setminus S = \{j_1, j_2, \cdots, j_l\}$ where $l = |T \setminus S|$. Note that

  $$R \subseteq S, R \cup \{j_1, j_2, \cdots, j_l\} = T, S \cup \{j_1, j_2, \cdots, j_l\} = S \cup T.$$

  Then, the following inequalities can be obtained by applying (4): (will be shown on the board)

- The relationship between (3) and (4) is similar to that between supermodularity and increasing differences (introduced in Advanced Noncooperative Game Theory).

- Let $\pi : N \to N$ be a one-to-one mapping, which is called a **permutation**. Let $i \in N$. Then, $\pi(i)$ is a number that denotes player $i$'s position in the ordering.

- Example: If $N = \{1, 2, 3\}$ and $\pi(1) = 2$, $\pi(2) = 3$ and $\pi(3) = 1$, then player 1 is second, player 2 is third, and player 3 is first:

  $$(3, 1, 2).$$

- For a player $i \in N$, define the set of players that precede $i$, $S^{\pi,i}$ by

  $$S^{\pi,i} = \{j \in N | \pi(j) < \pi(i)\} \quad (5)$$

  Define the vector $a^\pi = (a_1^\pi, a_2^\pi, \cdots, a_n^\pi)$ by

  $$a_i^\pi = v(S^{\pi,i} \cup \{i\}) - v(S^{\pi,i}) \quad (6)$$

  The value $a_i^\pi$ represents $i$'s marginal contribution in the permutation $\pi$.

**Theorem** Let $(N, v)$ be a convex game. For any permutation $\pi$, $a^\pi \in \mathcal{C}(N, v)$.

Sketch of Proof:

- It can be easily checked that $a^\pi$ is an imputation, with individual rationality following from superadditivity (which is implied by convexity).

- It remains to show coalitional rationality. Let $S \subseteq N$ be any coalition. It is sufficient to show that
$$\sum_{i \in N \setminus S} a_i^\pi \leq v(N) - v(S).$$
Let $N \setminus S = \{i_1, i_2, \cdots, i_l\}$ where $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_l)$. Then, (the rest will be filled in the lecture).

VII. The Bondareva-Shapley Theorem – Necessary and Sufficient Conditions for Nonemptiness of the Core

- First consider the following linear programming (LP) problem.

**(P)** Find $x \in \mathcal{R}^n$ that solves the following.
$$\min \sum_{i \in N} x_i$$
subject to
$$\sum_{i \in S} x_i \geq v(S), \ \forall S \subseteq N, S \neq \emptyset$$

- Let $x^*$ be a solution to (P). $\mathcal{C}(N, v) \neq \emptyset \Leftrightarrow \sum_{i \in N} x_i^* \leq v(N)$. (Actually, by the above constraint for $S = N$, $\sum_{i \in N} x_i^* = v(N)$.) $\rightarrow$ need a condition such that the statement in red holds. Use a result from linear programming.

VIII. Mathematical Aside: Linear Programming

- A linear programming (LP) problem is an optimization problem such that
  - the objective function (function that is to be maximized or minimzed) is linear in the decision variables

– the constraints are linear (in)equalities in the decision variables

**Primal Problem (P)**: Choose $x \in \mathcal{R}^n$ that solves the following problem.

$$\min \sum_{i=1}^{n} c_i x_i \tag{7}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_j, \;\; j = 1, 2, \cdots, m \tag{8}$$

$$x_i \geq 0, \;\; i = 1, 2, \cdots, n. \tag{9}$$

- To analyze the original problem, called the **primal**, it is sometimes useful to solve the **dual** problem, which is defined in the following.

**Dual Problem (D)**: Choose $y \in \mathcal{R}^m$ that solves the following maximization problem.

$$\max \sum_{j=1}^{m} b_j y_j \tag{10}$$

subject to

$$\sum_{j=1}^{m} a_{ij} y_j \leq c_i, \;\; i = 1, 2, \cdots, n \tag{11}$$

$$y_j \geq 0, \;\; j = 1, 2, \cdots, m. \tag{12}$$

- The primal problem is said to be **feasible** if there exists $x$ that satisfy (8)-(9). Likewise, the dual problem is **feasible** if there exists $y$ and (11)-(12). Such $x$ and $y$ are called **feasible vectors** or **feasible solutions**.
- The problem (P) is said to be **infeasible** if there is no feasible solution.
- The problem (P) is said to be **unbounded** if for every real number $K$, there is a feasible $x$, such that

$$\sum_{i=1}^{n} c_i x_i < K$$

- Given a linear program, there are only three possibilities.
  - The problem has an optimal solution.
  - The problem is infeasible.
  - The problem is unbounded.

  (One possibility is ruled out: no solution but not unbounded. This is a fact that does not hold for general (nonlinear) optimization problems.)

**Weak Duality Theorem.** Let $x \in \mathcal{R}^n$ and $y \in \mathcal{R}^m$ be arbitrary vectors that are feasible to (P) and (D) respectively. Then, the following inequality holds.

$$\sum_{j=1}^{m} b_j y_j \leq \sum_{i=1}^{n} c_i x_i \tag{13}$$

Specifically, if $x^*$ solves (P) and $y^*$ solves (D), then

$$\sum_{j=1}^{m} b_j y_j^* \leq \sum_{i=1}^{n} c_i x_i^* \tag{14}$$

- Immediate consequences:
  - (P) unbounded $\Rightarrow$ (D) infeasible.
  - (D) unbounded $\Rightarrow$ (P) infeasible.
- A stronger result can be obtained.

**(Strong) Duality Theorem.** If there is a solution $x^*$ to the problem (P), then there is a solution $y^*$ to the dual problem (D) and the following equality holds.

$$\sum_{i=1}^{n} c_i x_i^* = \sum_{j=1}^{m} b_j y_j^* \tag{15}$$

- Finally, consider a problem without a nonnegativity constraint (9):

$$\min \sum_{i=1}^{n} c_i x_i \tag{16}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_j, \;\; j = 1, 2, \cdots, m \tag{17}$$

- The problem above can be formulated in the form of (7)-(9) by defining two non-negative variables $x_i'$ and $x_i''$ such that

$$x_i = x_i' - x_i''$$

- The dual of (16) is the following:

$$\max \sum_{j=1}^{m} b_j y_j \tag{18}$$

subject to

$$\sum_{j=1}^{m} a_{ij} y_j = c_i, \;\; i = 1, 2, \cdots, n \tag{19}$$

$$y_j \geq 0, \;\; j = 1, 2, \cdots, m. \tag{20}$$

IX. Back to the Theorem

- Recall now the original problem.

**(P)** Find $x \in \mathcal{R}^n$ that solves the following.

$$\min \sum_{i \in N} x_i \tag{21}$$

subject to

$$\sum_{i \in S} x_i \geq v(S) \; \forall S \subseteq N, S \neq \emptyset. \tag{22}$$

- Consider the dual of (P), which is given in the following.

**(D)** Find $(\delta_S)_{\emptyset \neq S \subseteq N}$ that solves the following.

$$\max \sum_{S \subseteq N, S \neq \emptyset} \delta_S v(S), \tag{23}$$

subject to

$$\sum_{S \subseteq N, i \in S} \delta_S = 1, \ \forall i \in N, \tag{24}$$

$$\delta_S \geq 0, \ \forall S \subseteq N, S \neq \emptyset. \tag{25}$$

- Duality theorem (see Appendix) implies that if the solution $\delta^* = (\delta^*_S)_{S \subseteq N, S \neq \emptyset}$ to the problem (D), then $\sum_{i \in N} x^*_i = \sum_{S \subseteq N, S \neq \emptyset} \delta^*_S v(S)$. Thus, for the core to be nonempty, it is necessary and sufficient for $\sum \delta^*_S v(S) \leq v(N)$.

- A collection of coalitions, $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, is said to be a **balanced family** if there exist weights $(\delta_S)_{S \in \mathcal{B}}$ such that

$$\sum_{S \in \mathcal{B}, i \in S} \delta_S = 1, \ \forall i \in N$$

- A game $(N, v)$ is said to be **balanced** if for every balanced family of coalitions $\mathcal{B}$ with nonnegative weights $(\delta_S)_{S \in \mathcal{B}}$,

$$\sum_{S \in \mathcal{B}} \delta_S v(S) \leq v(N).$$

**Bondareva-Shapley Theorem (weak version).**

$$\mathcal{C}(N, v) \neq \emptyset \Leftrightarrow (N, v) \text{ is balanced.}$$

- $(N, v)$ being balanced required checking the inequality for all balanced families $\mathcal{B}$. It can be shown that the theorem holds even when considering minimal balanced collections.

- A balanced family $\mathcal{B}$ is a **minimal balanced family** if there is no balanced family $\mathcal{B}'$ with $\mathcal{B}' \subsetneq \mathcal{B}$.

**Bondareva-Shapley Theorem (strong version).**

$$\mathcal{C}(N,v) \neq \emptyset \Leftrightarrow \text{For every } \underline{\text{minimal}} \text{ balanced family } \mathcal{B},$$

$$\sum_{S \in \mathcal{B}} \delta_S v(S) \leq v(N).$$