The Bargaining Problem and Bargaining Solutions

I. Overview

- First model: cooperative game with two players $\rightarrow$ bargaining game
- Abstract mathematical model representing a bargaining situation

II. Nash Bargaining Problem

- A bargaining problem is defined by two components (B, d) where
  - B, the **feasible set**, represents the set of payoffs that can be achieved by the two players. It is assumed that $B \subset \mathcal{R}^2$.
  - $d = (d_1, d_2)$, the **disagreement point**, represents the outcome when bargaining fails. It is assumed that $d \in B$.
- **Mathematical assumptions**
  - $B \subset \mathcal{R}^2$ is a compact and convex set.
  - There exists $u = (u_1, u_2) \in B$ such that $u_1 > d_1$ and $u_2 > d_2$.
- <u>Notation</u>: Let $\tilde{B} = \{(u_1, u_2) \in B | u_1 \geq d_1, u_2 \geq d_2\}$.
- <u>Nash's solution</u>: For each (B, d), choose (u_1^*, u_2^*) that solves the following maximization problem:

$$\max_{(u_1,u_2)\in\tilde{B}} (u_1 - d_1)(u_2 - d_2) \tag{1}$$

III. Key Definitions and Results from Mathematics

- A set $X \subset \mathcal{R}^n$ is **closed** $\Leftrightarrow$ for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$ such that $x_k \rightarrow x$, then $x \in X$.
- A set $X \subset \mathcal{R}^n$ is **bounded** $\Leftrightarrow$ there exists M such that $|x_i| \leq M$ for every $x = (x_1, x_2, \cdots, x_n) \in X$ and $i = 1, 2, \cdots, n$
- A set $X \subset \mathcal{R}^n$ is **compact** $\Leftrightarrow$ X is both closed and bounded.
- A set $X \subset \mathcal{R}^n$ is **convex** $\Leftrightarrow$ for every $x, x' \in X$ and $\lambda \in [0, 1]$, $(1 - \lambda)x + \lambda x' \in X$.
- For $x, y \in \mathcal{R}^n$, define the **distance** between x and y by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

- Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{R}^n$. The sequence $\{x_k\}_{k=1}^{\infty}$ is said to **converge to** x (denoted by $x_k \to x$) if for every $\epsilon > 0$, there exists a number N such that for all $n \geq N$, $d(x_n, x) < \epsilon$. x is said to be the **limit** of the sequence $\{x_k\}_{k=1}^{\infty}$ and is denoted by $\lim_{k\to\infty} x_k = x$.

- Let $\{x_k\}$ and $\{y_k\}$ be two sequences such that $x_k \to x$ and $y_k \to y$.
  - For any real numbers α and β, $\alpha x_k + \beta y_k \to \alpha x + \beta y$.
  - If $x_k \leq y_k$ for all k, then $x \leq y$.

- Let $f : X \to K$ be a function where $X \subset \mathcal{R}^n$ and $K \subset \mathcal{R}^m$ are compact. f is said to be **continuous** if for every sequence $\{x_k\}_{k=1}^{\infty} \subset X$, $x_k \to x \Rightarrow f(x_k) \to f(x)$. That is, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ converges to $f(x)$.

Weierstrauss' Theorem. Let $K \subset \mathcal{R}^n$ be a compact set and $f : K \to \mathcal{R}$ a continuous function. Then, there exists $x^* \in K$ such that $f(x) \leq f(x^*)$ $\forall x \in K$. That is, the maximization problem

$$\max_{x \in K} f(x)$$

has at least one solution. The statement also holds when "max" is replaced by "min."

IV. Nash Bargaining Solution and Four Axioms

- $\mathcal{B}$: the set of all bargaining problems

- A **bargaining solution** is a function $f : \mathcal{B} \to \mathcal{R}^2$ such that for each $(B, d) \in \mathcal{B}$, $f(B, d) \in B$.

- Notation: $f_i(B, d)$ denotes the ith component of $f(B, d)$ $(i = 1, 2)$

- To justify his bargaining solution, Nash showed that
  - Nash bargaining solution satisfies four nice properties or **axioms**. (These will be explained in the following.)
  - Nash bargaining solution is the only bargaining solution that satisfies these axioms.

Pareto Efficiency (PE): A bargaining solution f satisfies **Pareto efficiency** if for each (B, d), there is no $(u_1', u_2') \in B$ such that $u_i' \geq f_i(B, d)$ for all $i \in \{1, 2\}$ and $u_j' > f_j(B, d)$ for some $j \in \{1, 2\}$.

- A bargaining problem (B, d) is said to be **symmetric** if
  - $(u_1, u_2) \in B \Leftrightarrow (u_2, u_1) \in B$
  - $d_1 = d_2$

Symmetry (SYM): A bargaining solution f satisfies **symmetry** if for every symmetric bargaining problem (B, d), $f_1(B, d) = f_2(B, d)$.

- Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be positive real numbers and $\beta = (\beta_1, \beta_2) \in \mathcal{R}^2$. Consider the new bargaining problem (B', d') where

$$B' = \{(u_1', u_2') \in \mathcal{R}^2 | u_1' = \alpha_1 u_1 + \beta_1, u_2' = \alpha_2 u_2 + \beta_2, (u_1, u_2) \in B\} \tag{2}$$
$$d' = (d_1', d_2') = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2) \tag{3}$$

- It can be checked that all the assumptions of the bargaining problem (B', d') are satisfied.

Covariance (COV): f satisfies **covariance** if for every bargaining problem (B', d') that are defined by (2) and (3),

$$f(B', d') = \big(f_1(B', d'), f_2(B', d')\big) = (\alpha_1 f_1(B, d) + \beta_1, \alpha_2 f_2(B, d) + \beta_2)$$

- Other terms of covariance: "Independence of Positive Affine Transformation," "Invariance with Respect to Affine Transformations"

Independence of Irrelevant Alternatives (IIA): f satisfies **independence of irrelevant alternatives** if for every bargaining problem (B, d) and $U \subseteq B$ such that $d \in U$ with $(U, d) \in \mathcal{B}$,

$$f(B, d) \in U \Rightarrow f(U, d) = f(B, d).$$

- The following is a theorem that provides a **characterization** of the Nash bargaining solution.

Theorem. The Nash bargaining solution is the unique bargaining solution that satisfies PE, SYM, COV, and IIA.

V. A Sketch of the Proof of the Theorem

1. Notation: $H(u_1, u_2) = (u_1 - d_1)(u_2 - d_2)$. Note that H is a continuous function of (u_1, u_2).

2. First, it needs to be established that for each bargaining problem (B, d), maximization problem (1) has a solution and is unique.

   - The existence part follows from Weierstrauss' Theorem – H is continuous and the set $\tilde{B} = \{(u_1, u_2) \in B | u_1 \geq d_1, u_2 \geq d_2\}$ is compact.
   - If there were two solutions – s^* and t^* with $s^* \neq t^*$ – to the maximization problem (1), then it can be shown that

   $$H\left(\frac{s_1^* + t_1^*}{2}, \frac{s_2^* + t_2^*}{2}\right) > H(s_1^*, s_2^*) = H(t_1^*, t_2^*) \tag{4}$$

   - Because $\tilde{B}$ is convex, $\left(\frac{s_1^*+t_1^*}{2}, \frac{s_2^*+t_2^*}{2}\right) \in \tilde{B}$, and equation (4) contradicts the definition of s^* and t^*.

3. Let f^N be a bargaining solution that assigns to each (B, d) the solution of (1). It can be checked that f^N satisfies the four axioms. Therefore, this proves the existence of a bargaining solution satisfying the four axioms.

4. To show that there is only one such bargaining solution, let g be a bargaining solution satisfying the four axioms. The objective is to show that for any $(B, d) \in \mathcal{B}$,

$$g(B, d) = f^N(B, d).$$

5. Take any (B, d) and denote by $u^* = f^N(B, d)$. Consider a positive affine transformation that transforms (B, d) to (B', d') such that $d' = (0, 0)$ and $u^* = (1/2, 1/2)$. (Find such α_i and β_i $i = 1, 2$.)

6. For any $(u_1', u_2') \in B'$, $u_1' + u_2' \leq 1$. (This part can be shown by way of contradiction and using that B' is a convex set.)

7. Because B' is bounded, there exists an isosceles right triangle that contains B'. Let T represent the area enclosed by the triangle including the boundary.

8. $(T, (0, 0))$ satisfies the conditions for a bargaining problem:
   - T is closed and bounded $\Rightarrow$ T is compact.
   - T is convex.
   - $(1/2, 1/2) \in T$ is a point that yields higher payoffs for both player than $d' = (0, 0)$.

9. $f^N (T, (0, 0)) = (1/2, 1/2)$ by direct calculation.

10. $g (T, (0, 0)) = (1/2, 1/2)$ by PE and SYM. Therefore, $g(T, (0, 0)) = f^N(T, (0, 0))$.

11. Because $(0, 0) \in B'$, $(1/2, 1/2) \in B'$, and $B' \subset T$ and f^N satisfies IIA (by 4.), $f^N(B', (0, 0)) = (1/2, 1/2)$. Similarly, becauge g satisfies IIA, $g(B', (0, 0)) = (1/2, 1/2)$.

12. By COV, $g(B, d) = (u_1^*, u_2^*) = f^N(B, d)$.

VI. Alternative Bargaining Solutions

- The fourth axiom (IIA) is not without controversy. (Example will be shown on the board.)
- An alternative axiom : monotonicity (MON)
- Kalai and Smorodinksy (1975) define the following solution and show that it is the only solution satisfying (PE), (SYM), (COV), and (MON) (to be defined later)

- Notation: For a bargaining solution (B, d), define the following

$$\bar{u}_i(B) = \max\{u_i | (u_i, u_j) \in B \text{ for some } u_j\}.$$

The point $(\bar{u}_1, \bar{u}_2)$ is called the **ideal point.**

- To state the monotonicity axiom, define another function g_i^B, defined on the payoff of the other player u_j such that

$$g_i^B(u_j) = \begin{cases} u_i & \text{if } (u_i, u_j) \in B \text{ is Pareto efficient} \\ \bar{u}_i(B) & \text{otherwise} \end{cases}$$

Monotonicity (MON): f satisfies **monotonicity** if for any two bargaining problems (B, d) and (B', d) such that $\bar{u}_j(B) = \bar{u}_j(B')$ and $g_i^B(u_j) \le g_i^{B'}(u_j)$ for all u_j, $f_i(B, d) \le f_i(B', d)$ holds.

- The **Kalai-Smorodinsky bargaining solution** is defined by the following procedure. First, draw a line between d and $\bar{u}(B) = (\bar{u}_1(B), \bar{u}_2(B))$. Then, find the point on this line such that it is in B and is Pareto efficient. This point is the Kalai-Smorodinsky solution of the problem (B, d). Let f^{KS} be the function, called the Kalai-Smorodinsky bargaining solution, such that $f^{KS}(B, d)$ is the Kalai-Smorodinsky solution of (B, d).

- f^{KS} is the unique solution that satisfies PE, SYM, COV, and MON. (Kalai and Smorodinksy (1975))

- Nash bargaining solution does not satisfy the following monotonicity condition.

- Another solution: **egalitarian solution** (Kalai (1977)). For (B, d), the egalitarian solution is a function f^E such that $f^E(B, d) = (u_1, u_2)$ that satisfies the following.

  1. $u_1 - d_1 = u_2 - d_2$
  2. There does not exist $(u_1', u_2') \in B$ such that $u_i' > u_i$ for all $i = 1, 2$.

VIII. Nash Bargaining Solution as Equilibrium Outcome of a Noncooperative Bargaining Game

- Nash's original game (Nash (1953))

  - Each player i reports u_i
  - If $(u_1, u_2) \in B$, then player i receives u_i. If not, each player i receives d_i.
  - Multiple Nash equilibria due to discontinuity in the payoff function.
  - "Smoothing" the payoff function and take limit – only one Nash equilibrium, which is the Nash bargaining solution

- Rubinstein's alternating offers model (Rubinstein (1982)) – now often used for the noncooperative rationale for Nash bargaining solution
  - Period 1: Player 1 offers $(u_1, u_2) \in B$ to player 2. Player 2 chooses whether to accept or reject this offer.
    * Player 2 accept $\rightarrow$ player 1's payoff u_1 and player 2's payoff is u_2
    * Player 2 reject $\rightarrow$ Period 2
  - Period 2: Player 2 offers $(u_1', u_2') \in B$ to player 1. Player 1 chooses whether to accept or reject.
    * Player 1 accept $\rightarrow$ player 1's payoff $\delta u_1'$ and player 2's payoff is $\delta u_2'$ where δ is the discount factor with $0 < \delta < 1$.
    * Player 1 reject $\rightarrow$ Period 3
  - It can be shown that any payoff outcome in the bargaining region B can be achieved via a Nash equilibrium. However, there is only one subgame-perfect equilibrium. The details of the results are shown below.

Theorem. There exists a unique subgame-perfect equilibrium of this game. This equilibrium satisfies the following.

1. The strategies of the players are stationary – they always propose the same proposal, and their condition to accepting or rejecting an offer is the same throughout.

2. In equilibrium, Player 1's proposal is accepted – there is no delay in bargaining.

3. Moreover, as $\delta \rightarrow 1$, the equilibrium payoffs converge to the Nash bargaining solution.

IX. Some Notes on the Literature

- The original model – Nash (1950)

- Nash's bargaining game and axioms – Nash (1953)
- Kalai-Smorodinsky solution – Kalai and Smorodinksy (1975)
- Egalitarian solution and other proportional solutions – Kalai (1977)

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