Application: Household's Decision Making in Continuous Time

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1 Household's Utility Maximization Problem

- Consider the following utility maximization problem of a household:

$$\begin{aligned} \max \quad & U_0 = \int_0^\infty \exp(-\rho t) u(c_t) dt \\ \text{s.t.} \quad & \dot{a}_t (\equiv da_t/dt) = ra_t + w_t - c_t \quad \text{(flow budget constraint)}, \\ & a_0 \text{ given} \quad \text{(initial condition)}, \\ & \lim_{t\to\infty} a_t \exp(-rt) \geq 0 \quad \text{(no-Ponzi-game condition)}, \end{aligned}$$

  where

  - a_t and c_t: amount of her assets (state) and consumption (control); a dot over a variable indicates a time derivative.
  - r and w_t: the interest– and wage rate, the former of which is assumed to be constant over time for simplicity,

    Assumption 1. *r is constant over time.*

  - $\rho > 0$: the subjective discount rate,
  - $u(c)$: the *instantaneous utility function* (瞬時効用関数), assumed to be $u' > 0$ and $u'' < 0$.

- The no-Ponzi-game condition (hereafter, NPG) ensures that the household does not asymptotically tend to a negative wealth. Without this condition, the household can increase her consumption by borrowing to such level that feasibility is violated.

- In the lecture note on 6/21, we have rigorously shown that the Euler equation and the transversality condition (TVC) are the sufficient conditions of the solutions under the concavity of u. (here the strict concavity is assumed).

- To *derive* these conditions, let us use the following cookbook procedure:

Step 0. Construct the following Lagrangian associated with the "finite-horizon counterpart" of the above problem:

$$\begin{aligned} L &= \int_0^T \left\{ e^{-\rho t} u(c_t) + \lambda_t [r a_t + w_t - c_t - \dot{a}_t] \right\} dt + \zeta a_T e^{-rT} \\ &= \int_0^T e^{-\rho t} \left\{ u(c_t) + \mu_t [r a_t + w_t - c_t - \dot{a}_t] \right\} dt + \zeta a_T e^{-rT}, \quad \mu_t = \lambda_t e^{\rho t}, \end{aligned}$$

L is further rewritten as follows:

$$L = \int_0^T e^{-\rho t} \left[H(a_t, c_t, \mu_t) + (\dot{\mu}_t - \rho \mu_t) a_t \right] dt - \mu_T a_T e^{-\rho T} + \mu_0 a_0 + \zeta a_T e^{-rT},$$

where H is called the *current–value Hamiltonian* (当該価値ハミルトニアン):

$$H(a_t, c_t, \mu_t) = u(c_t) + \mu_t (r a_t + w_t - c_t).$$

Step 1. Derive the first-order-conditions:

$$c_t : \quad \partial H / \partial c_t = 0, \tag{1}$$

$$a_t : \quad \partial H / \partial a_t + \dot{\mu}_t - \rho \mu_t = 0, \tag{2}$$

$$a_T : \quad \mu_T e^{-\rho T} = \zeta e^{-rT}, \tag{3}$$

$$\zeta : \quad a_T e^{-rT} \geq 0, \ \zeta a_T e^{-rT} = 0. \tag{4}$$

Step 2. Make simpler expressions:

$$\partial H / \partial c_t = 0 \ \Leftrightarrow u'(c_t) = \mu_t, \tag{5}$$

$$\partial H / \partial a_t + \dot{\mu}_t - \rho \mu_t = 0 \Leftrightarrow \dot{\mu}_t / \mu_t = \rho - r, \tag{6}$$

$$\mu_T a_T e^{-\rho T} = 0. \tag{7}$$

Step 3. Take the limit of $T \to \infty$ in (5)–(7).

- Differentiating both sides of (5) with respect to time,

$$\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = \frac{\dot{\mu}_t}{\mu_t}. \tag{8}$$

- Substituting (8) into (6) yields:

$$-\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{9}$$

(6), or (9) is called the *Euler equation.* In the context of Macroeconomics, this is also called *Keynes–Ramsey Rule.*

2 Some Features

2.1 Economic Implications of Euler Equation

- What does the Euler equation provide us?

  → Suppose that the household decreases c_t but increases $c_{t+\Delta t}$ with U_0 unchanged.

  Caution: Rigorously, in continuous time models we can not change the variable at an instant of time. However, short-cuts like this do lead usable results.

- By differentiating the life-time utility, and imposing $dU_0 = 0$, we have the marginal rate of substitution (MRS) of consumption at t for $t + \Delta t$:

$$\begin{aligned} dU_0 = 0 &\Rightarrow u'(c_t)dc_t + e^{-\rho\Delta t}u'(c_{t+\Delta t})dc_{t+\Delta t} \\ &\Rightarrow -\frac{dc_{t+\Delta t}}{dc_t} = \frac{u'(c_t)}{e^{-\rho\Delta t}u'(c_{t+\Delta t})}. \end{aligned}$$

- In analogy with a two-period utility maximization problem, the above MRS must be equal to the gross interest rate:

$$\begin{aligned} \frac{u'(c_t)}{e^{-\rho\Delta t}u'(c_{t+\Delta t})} &= 1 + r\Delta t \\ \Rightarrow \quad \frac{1}{\Delta t}\frac{u'(c_t) - e^{-\rho\Delta t}u'(c_{t+\Delta t})}{e^{-\rho\Delta t}u'(c_{t+\Delta t})} &= r. \end{aligned} \tag{10}$$

- Taking the limit $\Delta t \to 0$, we obtain

$$\rho - \frac{c_t u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} = r. \tag{11}$$

  This equation (11) is exactly the Euler equation.

2.2 How does consumption change over time?

- Euler equation (9), or (11) implies that consumption increases or decreases over time depending on whether the interest rate exceeds or is less than the subjective discount rate: Given $c_t > 0$,

$$\dot{c}_t \gtreqless 0 \;\Leftrightarrow\; r \gtreqless \rho,$$

  which comes from the fact that $-cu''/u' > 0$ as long as $c > 0$.

  → The sign of gap $r - \rho$ determines whether or not consumption grows over time.

Proposition 1. *Suppose that $c_t > 0$ for all $t \geq 0$. Consumption increases (decreases) over time if and only if $r > (<)\rho$, and remains constant if and only if $r = \rho$.*

- On the other hand, $-cu''/u' > 0$, which is the elasticity of marginal utility, determines steepness of the slope of consumption:

$$\frac{\dot{c}_t}{c_t} = \left(-\frac{c_t u''(c_t)}{u'(c_t)}\right)^{-1} (r - \rho).$$

  $\to$ $(-cu''/u')^{-1}$ is therefore called the degree of *intertemporal elasticity of substitution (IES, 異時点間の代替の弾力性)*.

Definition 1 (Intertemporal Elasticity of Substitution between t and $t + \Delta t$)**.**

$$\sigma(t, t + \Delta t) = \frac{d \log(c_{t+\Delta t}/c_t)}{d \log(1 + r\Delta t)}.$$

Lemma 1. *Under the situation (10) holds,* $\lim_{\Delta t \to 0} \sigma(t, t + \Delta t) = \left(-\frac{c_t u''(c_t)}{u'(c_t)}\right)^{-1}$.

Proof. Exercise. □

Proposition 2. *The growth rate of consumption becomes higher if the value of IES becomes higher.*

2.3 Difference between TVC and NPG

- Note that the household can not only lend to other households, but also borrow from them. This directly means a_t can be negative.
- Therefore, in the absence of any restrictions on borrowing, the solution to the maximization problem is a trivial one: it is optimal for the households to accumulate debts and to maintain a level of consumption such that the marginal utility of consumption equals zero.

  $\to$ NPG is the constraint that prohibits the household to do so.
- On the other hand, the TVC in this problem can be also expressed as

$$\lim_{t \to \infty} a_t \exp(-rt) = 0. \tag{12}$$

- So do not confuse the NPG and TVC:
  - NPG: the constraint that prohibits the household to default on sovereign debt, whereas
  - TVC: the condition for utility maximization under NPG is imposed.

3 Consumption Behaviors

Then, how does the household decide the optimal paths of consumption and assets?

3.1 Intertemporal Budget Constraint

- The flow budget constraint of the household is expressed as

$$\dot{a}_t - ra_t = w_t - c_t.$$

- Multiplying both sides by e^{-rt} and integrating the resulting equation from 0 to ∞,

$$\lim_{t\to\infty} a_t \exp(-rt) - a_0 = \int_0^{\infty} w_t e^{-rt} dt - \int_0^{\infty} c_t e^{-rt} dt. \tag{13}$$

- (13) is the *intertemporal budget constraint* (異時点間の予算制約式). This equation gives us the following two important features of the model:

  1. The household can not plan without anticipating the entire path of both w_t and r.
     $\rightarrow$ Expectations are crucial !
  2. e^{-rt} can be viewed as the price of the good at date t.

- The TVC shows that the first term on the left-hand-side of (13) is zero:

$$\int_0^{\infty} c_t e^{-rt} dt = a_0 + \int_0^{\infty} w_t e^{-rt} dt. \tag{14}$$

3.2 Consumption Function

- The instantaneous utility u is often specified as

$$u(c) = \begin{cases} \dfrac{c^{1-\theta} - 1}{1-\theta} & \text{for } \theta > 0, \theta \neq 1, \\ \ln c & \text{for } \theta = 1. \end{cases} \tag{15}$$

- This function is called the *CRRA utility function*, where CRRA is the abbreviated name of the "Constant Relative Risk Aversion." The degree of relative risk aversion is defined as

$$\text{degree of RRA} = -\frac{cu''(c)}{u'(c)}$$

  $\rightarrow$ This becomes constant, θ, if u is specified as (15).

  $\rightarrow$ The value of intertemporal elasticity of substitution also becomes constant, $1/\theta$.

- Then, the Euler equation (9) becomes

$$\dot{c}_t / c_t = \theta^{-1}(r - \rho). \tag{16}$$

  $\rightarrow$ By using the method of separation of variables, we easily have

$$c_t = c_0 \exp\left[\theta^{-1}(r-\rho)t\right], \tag{17}$$

- Substituting (17) into (14) yields

$$c_0 \int_0^\infty \exp\left[\left(\frac{(1-\theta)r-\rho}{\theta}\right)t\right]dt = a_0 + \int_0^\infty w_t e^{-rt}dt. \tag{18}$$

Assumption 2. $(1-\theta)r < \rho$.

- Then, we can solve (18) for c_0:

$$c_0 = \frac{\rho-(1-\theta)r}{\theta}\left(a_0 + \int_0^\infty w_t e^{-rt}dt\right).$$

- It is easily verified that the consumption function given above applies for all t :

$$c_t = \frac{\rho-(1-\theta)r}{\theta}\left(a_t + \int_t^\infty w_\tau e^{-r(\tau-t)}d\tau\right). \tag{19}$$

  (19) gives the consumption function of the dynamically optimizing household.

  → What factors affect consumption? How?