Infinite-horizon Problems in Discrete Time

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This lecture note is mainly based on Ch. 5 of Sorger (2015) and Ch. 4 of Stokey, Lucas and Prescott (1989). For more general explanations of discrete-time infinite-horizon dynamic programming technique in economics, see these two books, Ch. 6 of Acemoglu (2009) and Ch. 12 of Sundaram (1996).

1 Model Formulation and Terminology

1.1 Two Kinds of Variables and Transition Equation

- *State variable* (状態変数): this is the kind of variables already determined at the beginning of each period. After the decision making of an individual, these variables change over time thorough the *transition equation* (推移方程式).

  Example: Suppose that you hold a_t units of financial assets today (i.e., the sum of your all assets: bank deposits, equities, bond, real estate...). This has been already determined by your past savings. You receive the interest income $r_t a_t$ and the wage income w_t. Your budget constraint is then given by

  $$\underbrace{a_{t+1} - a_t}_{\text{savings}} = \underbrace{r_t a_t + w_t}_{\text{income flow}} - \underbrace{c_t}_{\text{consumption}}.$$

  Then, once you decide c_t, your assets tomorrow, a_{t+1}, is accordingly determined. Thus the budget constraint works as the transition equation with respect to your assets.

- *Control variable* (制御変数): this is a variable immediately under control. In the above example, consumption corresponds to this.

- Hereafter, let $x_t \in X \subseteq \mathbb{R}_+$ be the state variable. X is sometimes called the *state space.* On the other hand, let $c_t \in C \subseteq \mathbb{R}_+$ denote the control variable.

- It is assumed that x_t evolves according to the following transition equation:

$$x_{t+1} = g(x_t, c_t, t),$$

where $g : X \times C \times \mathbb{N}_0 \to X$ is the *transition function* (推移関数) and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the set of nonnegative integers.

1.2 Feasible Path

- Remember the above example of the household budget constraint. Suppose that you can not borrow any funds from others. If this is the case, your consumption in period t must satisfy

$$0 \leq c_t \leq (1 + r_t)a_t + w_t.$$

That is, your consumption plan in t is *feasible* if $c_t \in [0, (1 + r_t)a_t + w_t]$ in this case.[1]

- Let us generalize the above notion. Let $\mathcal{G}(x_t, t)$ denote the set of feasible values of c_t.

Then, we can define the *feasible path* (実行可能経路).

Definition 1. *Given $x_0 \in X$, $\{x_t, c_t\}_{t=0}^{\infty}$ is the feasible path if*

$$(x_{t+1}, c_t) \in \Omega(x_t, t) \equiv \{(x', c) \in X \times C \mid c \in \mathcal{G}(x_t, t), x' = g(x_t, c, t)\} \forall t \in \mathbb{N}_0.$$

Hereafter, we omit the argument "t" of $g(x_t, c_t, t)$, $\mathcal{G}(x_t, t)$ and $\Omega(x_t, t)$ unless to do so would cause confusions.

1.3 Objective Function

- Throughout this note, the objective function is

$$J(\{c_t\}_{t=0}^{\infty}) = \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t u(c_t), \ \ \beta \in (0, 1),$$

where $u(c)$ is the one-period return function. Hereafter we simply express J as $\sum_{t=0}^{\infty} \beta^t u(c_t)$.

- It is assumed that J is bounded.

In sum, the infinite-horizon discounted problem in discrete time is formulated as

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & (x_{t+1}, c_t) \in \Omega(x_t), \quad t = 0, 1, 2, \dots \\ & x_0 \in X \text{ given} \end{aligned} \tag{P}$$

[1] Caution: In our actual economic life, we can have debts. So, c_t can *temporally* exceeds $(1 + r_t)a_t + w_t$, which results in the negative value of a_{t+1}. Off course, the debt must be paid off in a future date. If we consider a finite-horizon problem, $a_{T+1} \geq 0$ corresponds to the condition which prohibits default. In an infinite-horizon problem, however, there is no explicit terminal date. So we need another condition. This issue is discussed in the analysis of households' intertemporal consumption-saving decision makings.

2 Solution Method 1: Application of Optimal Control in Discrete Time

- Let us call the feasible path which is the solution to (P), the *optimal path* (最適経路).
- If we assume the smoothness and the concavity on u and g, we can use the method of optimal control in discrete time.

Assumption 1. *u and g are*

1. *continuously differentiable, and*
2. *concave.*

- Let $g_j(x, c) = \frac{\partial g(x,c)}{\partial j}$, $j = x, c$. The two properties in the above assumption jointly mean

$$u(c) - u(\tilde{c}) \geq u'(c)(c - \tilde{c}) \ \forall c, \tilde{c} \in C, \tag{1}$$

$$g(x, c) - g(\tilde{x}, \tilde{c}) \geq g_x(x, c)(x - \tilde{x}) + g_c(x, c)(c - \tilde{c}) \ \forall (x, c), (\tilde{x}, \tilde{c}) \in X \times C. \tag{2}$$

Theorem 1 (Sufficient Condition of the Optimal Path)**.** *Suppose that u and g satisfy Assumption 1. If a feasible path $\{x_t, c_t\}_{t=0}^{\infty}$ and $\{\lambda_t\}_{t=0}^{\infty}$ ($\lambda_t \geq 0$) satisfy*

$$u'(c_t) + \lambda_t g_c(x_t, c_t) = 0 \ \forall t \in \mathbb{N}_0, \tag{3}$$

$$\lambda_t - \beta g_x(x_{t+1}, c_{t+1})\lambda_{t+1} = 0 \ \forall t \in \mathbb{N}_0, \tag{4}$$

$$\lim_{t\to\infty} \lambda_t \beta^t x_{t+1} = 0, \tag{5}$$

then it follows that $\{x_t, c_t\}_{t=0}^{\infty}$ is an optimal path.

Proof. **Exercise 1:** Show this theorem using (1) and (2). □

- From (3) and (4), we have

$$u'(c_t) = \beta u'(c_{t+1}) \frac{g_c(x_t, c_t)}{g_c(x_{t+1}, c_{t+1})} g_x(x_{t+1}, c_{t+1}). \tag{6}$$

  (4) or (6) is called the *Euler equation.*

- On the other hand, substituting (3) into (5) yields

$$\lim_{t\to\infty} \beta^t \frac{u'(c_t)}{g_c(x_t, c_t)} x_{t+1} = 0. \tag{7}$$

  (5) or (7) is called the *transversality condition (hereafter, TVC).*

2.1 "Heuristic" Derivations of the Euler Equation and TVC

- How do we derive the Euler equation and TVC? → At first consider the following finite-horizon problem:

$$\begin{aligned}
\max \quad & \sum_{t=0}^{T} \beta^t u(c_t) \\
\text{s.t.} \quad & (x_{t+1}, c_t) \in \Omega(x_t), \ t = 0, 1, 2, \dots T, \\
& x_0 \text{ given}, \ x_{T+1} \geq 0.
\end{aligned}$$

- The above problem is the finite dimensional optimization. So we can simply look at the first-order conditions. The Lagrangian associated with this problem is given by

$$L = \sum_{t=0}^{T} \beta^t \Big[u(c_t) + \lambda_t (g(x_t, c_t) - x_{t+1}) \Big] + \mu x_{T+1},$$

  and the first-order-conditions are

$$\partial L / \partial c_t = 0: \quad u'(c_t) + \lambda_t g_c(x_t, c_t) = 0 \quad \text{for } t = 0, 1, 2, \dots, T, \tag{8}$$

$$\partial L / \partial x_{t+1} = 0: \quad \lambda_t = \beta g_x(x_{t+1}, c_{t+1}) \lambda_{t+1} \quad \text{for } t = 0, 1, 2, \dots, T-1, \tag{9}$$

$$\partial L / \partial x_{T+1} = 0: \quad \beta^T \lambda_T = \mu, \tag{10}$$

$$\partial L / \partial \mu = 0: \quad x_{T+1} \geq 0, \mu \geq 0, \mu x_{T+1} = 0. \tag{11}$$

- Substituting (8) into (9), we have

$$u'(c_t) = \frac{g_c(x_t, c_t)}{g_c(x_{t+1}, c_{t+1})} g_x(x_{t+1}, c_{t+1}) \beta u'(c_{t+1}) \quad \text{for } t = 0, 1, 2, \dots, T. \tag{12}$$

  On the other hand, substituting (10) into the third condition of (11) yields $\beta^T \lambda_T x_{T+1} = 0$. Substituting (8) into this expression, we obtain

$$\beta^T \frac{u'(c_T)}{g_c(x_T, c_T)} x_{T+1} = 0. \tag{13}$$

- Then, taking the limit $T \to \infty$ in (12) and (13), we can obtain (6) and (7).

Caution: On the necessities of the Euler equation and TVC, here we have to note the following two things:

1. In the above finite-horizon problem, the conditions (12) and (13) give the *necessary condition* of the optimization. Moreover, the sufficiency result in Theorem 1 holds true also in the finite-horizon problem. These two results imply that as long as we consider the finite-horizon problem, the Euler equation and the TVC are both necessary and sufficient condition of the optimization if the concavity and smoothness of u and g are ensured.

2. Consider the following "infinite-horizon counterpart" of L:

$$L_\infty = \lim_{T\to\infty} \left\{ \sum_{t=0}^{T} \beta^t \Big[u(c_t) + \lambda_t (g(x_t, c_t) - x_{t+1}) \Big] + \mu x_{T+1} \right\}$$

   Then, we can find that (8) and (9) are supported as the first-order-conditions even when $T \to \infty$. This means that the Euler equation is the necessary condition even in the infinite-horizon problem. However, it is not clear whether or not the TVC in the form of (7) is necessary in the infinite-horizon problem. This is because we can no longer obtain the conditions such as (10) or (11) when $T \to \infty$. The necessity of the TVC is therefore a difficult issue.

Then, in this note we assume that the TVC in the form of (7) is also necessary for the optimization in the infinite-horizon problem.

2.2 Example: The Ramsey Model in Discrete Time

- Consider the following problem:

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \forall t \in \mathbb{N}_0 \\ & k_0 > 0, \text{ given} \end{aligned}$$

  where $c_t \geq 0$ is consumption here, $k_t \geq 0$ is amount of physical capital, and $\delta \in (0,1)$ is the depreciation rate.

- It is assumed that both of u and f are increasing, concave and continuously differentiable.

Exercise 2: Derive the Euler equation and the TVC in the above problem.

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3 Solution Method 2: Dynamic Programming

- The "optimal control" approach is a very powerful tool if the smoothness and concavity of u and g are ensured.
- Unfortunately, we sometimes face
  1. Discrete choice problems
  2. u or g are not concave
- The dynamic programming does not require such assumptions.

3.1 Value Function

- Hereafter, we relax the smoothness and concavity given in Assumption 1.

Example: In the discrete-time Ramsey model in Section 2.2, g and $\mathcal{G}$ are respectively given by $g(k_t, c_t) = f(k_t) + (1-\delta)k_t - c_t$ and $\mathcal{G}(k_t) = [0, f(k_t) + (1-\delta)k_t]$.

- Briefly speaking, the *value function* (価値関数) is the maximized objective function, defined as follows

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Definition 2 (Value function). *The value function $V^* : X \to \mathbb{R}$ is defined as*

$$V^*(x_0) = \max_{\{x_t, c_t\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \;\middle|\; (x_{t+1}, c_t) \in \Omega(x_t) \forall t \in \mathbb{N}_0 \right\},$$

- In the discrete-time Ramsey model in Section 2.2, the value function is given by

$$V^*(k_0) = \max_{\{k_t, c_t\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \;\middle|\; \underbrace{k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, k_t \geq 0, c_t \geq 0}_{\Leftrightarrow (k_{t+1}, c_t) \in \Omega(k_t)} \forall t \in \mathbb{N}_0 \right\}$$

By a close look at the above equation, we can find that $V^*(k_0)$ is expressed also as

$$V^*(k_0) = \max_{\{k_t\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \;\middle|\; k_{t+1} \in [0, f(k_t) + (1-\delta)k_t] \forall t \in \mathbb{N}_0 \right\},$$

where $F(k_t, k_{t+1}) \equiv u(f(k_t) + (1-\delta)k_t - k_{t+1})$.

- Hereafter we assume that the value function $V^*(x_0)$ in Definition 2 can be also expressed as

$$V^*(x_0) = \max_{\{x_t\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \;\middle|\; x_{t+1} \in \Gamma(x_t) \forall t \in \mathbb{N}_0 \right\}, \tag{14}$$

where $\Gamma(x_t)$ is the feasible set of x_{t+1} (given by $[0, f(k_t)+(1-\delta)k_t]$ in the above example).

3.2 Principle of Optimality and Bellman Equation

- Given $x_0 \in X$, let $\{\hat{x}_t, \hat{c}_t\}_{t=0}^{\infty}$ is the optimal path ($\hat{x}_0 = x_0$). By its definition,

$$\begin{aligned} V^*(x_0) &= \sum_{t=0}^{\infty} \beta^t F(\hat{x}_t, \hat{x}_{t+1}) \\ &\geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \; \forall \{x_t\}_{t=1}^{\infty} \in \Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} | x_{t+1} \in \Gamma(x_t) \forall t \in \mathbb{N}_0\} \end{aligned}$$

- $\Pi(x_0)$ is the set of feasible paths of $\{x_t\}_{t=1}^{\infty}$.

Theorem 2. *Consider the problem in period $\tau > 0$:*

$$\begin{aligned} \max_{\{x_t\}_{t=\tau+1}^{\infty}} \quad & \sum_{t=\tau}^{\infty} \beta^{t-\tau} F(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in \Gamma(x_t) \forall t \geq \tau, \\ & x_\tau = \hat{x}_\tau. \end{aligned}$$

Then, the optimal path of the problem is $\{\hat{x}_t\}_{t=\tau}^{\infty}$.

Proof. Suppose otherwise. Specifically, suppose that the optimal path is given by $\{\tilde{x}_t\}_{t=\tau}^{\infty}$. Then,

$$\sum_{t=\tau}^{\infty} \beta^{t-\tau} F(\tilde{x}_t, \tilde{x}_{t+1}) \geq \sum_{t=\tau}^{\infty} \beta^{t-\tau} F(\hat{x}_t, \hat{x}_{t+1}) \Leftrightarrow \sum_{t=\tau}^{\infty} \beta^{t} F(\tilde{x}_t, \tilde{x}_{t+1}) \geq \sum_{t=\tau}^{\infty} \beta^{t} F(\hat{x}_t, \hat{x}_{t+1})$$

From the initial condition, $\tilde{x}_\tau = \hat{x}_t$. Then, adding $\sum_{t=0}^{\tau-1} \beta^t F(\hat{x}_t, \hat{x}_{t+1})$ to the both sides, we have

$$\sum_{t=0}^{\tau-1} \beta^t F(\hat{x}_t, \hat{x}_{t+1}) + \sum_{t=\tau}^{\infty} \beta^t F(\tilde{x}_t, \tilde{x}_{t+1}) \geq V^*(x_0),$$

which contradicts to the assumption that $\{\hat{x}_t\}_{t=0}^{\infty}$ is the optimal path in the problem in period 0. □

- Theorem 2 is calle the *Principle of Optimality.*
- Briefly speaking, the dynamic programming is the technique to *directly* obtain V^* by using this principle of optimality, rather than deriving the optimal path at first.
- At the heart of this approach to the optimization is the *Bellman equation* defined as follows:

Definition 3 (Bellman equation)**.** *The following functional equation is called the Bellman equation:*

$$V(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta V(y) \right\}. \tag{15}$$

3.3 Four Important Theorems

Theorem 3. *The value function V^* defined in (14) satisfies the Bellman equation (15).*

Proof. Omitted. See Theorem 4.2 of Stokey and Lucas (1989) or Theorem 5.7 of Sorger (2015). □

Theorem 4. *Suppose that the function $\hat{V}$ is the solution to the Bellman equation (15) and*

$$\lim_{t\to\infty} \beta^t \hat{V}(x_t) = 0 \ \forall \{x_t\}_{t=1}^{\infty} \in \Pi(x_0),$$

then $\hat{V} = V^*$ in (14).

Proof. Omitted. See Theorem 4.3 of Stokey and Lucas (1989) or Theorem 5.8 of Sorger (2015) □

- In Theorems 3–4, we have established a link between the value function V^* and the solution to the Bellman equation.

  → We can focus on the Bellman equation (15) instead of the original problem (P).

- Now we use V^* to characterize the optimal paths.

Theorem 5 (Necessary Condition of the Optimal Path)**.** *Suppose that $\{\hat{x}_t\}_{t=0}^{\infty}$ is the optimal path given $\hat{x}_0 = x_0 \in X$. Then, $V^*(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta V^*(\hat{x}_{t+1})$ for all $t \in \mathbb{N}_0$.*

Proof. At first we will show that $V^*(\hat{x}_t) = \sum_{n=t}^{\infty} \beta^{n-t} F(\hat{x}_n, \hat{x}_{n+1})$ for all $t \in \mathbb{N}_0$. The proof is by induction.

1. For $t = 0$, this is straightforward from the definition of $V^*(x_0)$ in (14).
2. Suppose that $V^*(\hat{x}_t) = \sum_{n=t}^{\infty} \beta^{n-t} F(\hat{x}_n, \hat{x}_{n+1})$ for a period, say, t. Then,

$$V^*(\hat{x}_t) = F(\hat{x}_t, \hat{x}_{t+1}) + \beta \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(\hat{x}_n, \hat{x}_{n+1}).$$

3. We will establish that it is also true for $t+1$. By the principle of optimality in Theorem (2),

$$V^*(\hat{x}_t) \geq F(\hat{x}_t, \hat{x}_{t+1}) + \beta \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(x_n, x_{n+1}) \quad (\text{where } x_{n+1} = \hat{x}_{n+1})$$

$$\Leftrightarrow \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(\hat{x}_n, \hat{x}_{n+1}) \geq \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(x_n, x_{n+1})$$

   for all $\{x_n\}_{n=t+2}^{\infty} \in \Pi(\hat{x}_{t+1})$. This implies that $V^*(\hat{x}_{t+1}) = \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(\hat{x}_n, \hat{x}_{n+1})$.

Thus, $V^*(\hat{x}_t) = \sum_{n=t}^{\infty} \beta^{n-t} F(\hat{x}_n, \hat{x}_{n+1})$ for all $t \in \mathbb{N}_0$. Then,

$$\begin{aligned} V^*(\hat{x}_t) &= F(\hat{x}_t, \hat{x}_{t+1}) + \beta \sum_{n=t+1}^{\infty} \beta^{n-(t+1)} F(\hat{x}_n, \hat{x}_{n+1}) \\ &= F(\hat{x}_t, \hat{x}_{t+1}) + \beta V^*(x^*_{t+1}). \end{aligned}$$

We have established the proof. □

- The next theorem provides a partial converse to Theorem 5.

Theorem 6 (Sufficient Condition of the Optimal Path). *Suppose that $\{x_t\}_{t=0}^{\infty}$ is the feasible path from x_0, i.e., $\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)$. Then, if $\{x_t\}_{t=0}^{\infty}$ satisfies*

$$V^*(x_t) = F(x_t, x_{t+1}) + \beta V^*(x_{t+1}), \tag{16}$$

and

$$\lim_{t \to \infty} \beta^t V^*(x_t) = 0, \tag{17}$$

then, $\{x_t\}_{t=0}^{\infty}$ is the optimal path.

Proof. From (16) ,

$$V^*(x_0) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^n V^*(x_{n+1})$$

Then, taking the limit $n \to \infty$ and using (17), we have $V^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$.

□

- Note that these theorems do not require that the optimization problem satisfies any concavity assumptions.

3.4 Policy Function

- We define the following function:

Definition 4 (Policy function). *$h : X \to X$ is called the policy function (政策関数) if*

$$h(x) = \arg\max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\}$$

- If we can obtain the value function V^* from the Bellman equation, we can restate the optimal path $\{\hat{x}_t\}_{t=1}^{\infty}$ in the following recursive form:

$$\forall x_0 \in X, \quad \hat{x}_{t+1} = h(\hat{x}_t), \quad t = 0, 1, 2, \dots \tag{18}$$

- In summary
  - Theorems 3 and 4 $\rightarrow V = V^*$ under the boundary condition $\lim_{n\to\infty} \beta^n V(x_n) = 0$. $\rightarrow$ We can focus on the Bellman equation (15) to obtain V^*, instead of the original problem (P).
  - Theorems 5 and 6 $\rightarrow$ Once V^* is obtained, the optimal path $\{\hat{x}_t\}_{t=0}^{\infty}$ is accordingly obtained by $\hat{x}_{t+1} = h(\hat{x}_t)$ and the boundary condition, $\lim_{n\to\infty} \beta^n V(\hat{x}_n) = 0$.
- Conversely, we *have to* obtain the value function from the Bellman equation.
- How?
  - Guess and verify
  - Value function iteration

    $\ddots$

3.5 Convergence of Value Function

- Given any V, define T by

$$T(V)(x) = \max_{x' \in \Gamma(x)} \{f(x, x') + \beta V(x')\}. \tag{19}$$

  T is called the Bellman operator.
  - $T : C(X) \rightarrow C(X)$, where $C(X)$ is a space of continuous function on X.
- At first, *arbitrarily* choose a function, say, $V_0(x) \in C(X)$, and substitute this into the right-hand-side of (19) for V.
- Then, in (19), the operator T gives the new function, say, $V_1(x)$.
- Substitute V_1 into the RHS of (19) for V.

  $\rightarrow$ the functional sequence, $\{V_j(x)\}_{j=0}^{\infty}$ is generated by the Bellman operator.
- Therefore, if $V_j(x)$ uniformly converges to $V^*(x)$, we can obtain the value function.

  ($*$) In Theorem 4.6 of Stokey and Lucas (1989, Ch. 4), it is shown that the operator $T : C(X) \rightarrow C(X)$ is a contraction mapping, which in turn shows that

$$T(V^*) = V^*, \quad \lim_{j\to\infty} T^j(V_0) = V^* \forall V_0 \in C(X).$$

  (Proof is omitted here) Then, V_j uniformly converges to V^*.

References

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[3] Stokey, N. L., R. E. Lucas Jr., and E. Prescott (1989) *Recursive Methods in Economic Dynamics*, Harvard University Press.

[4] Sundaram, R. K. (1996) *A First Course in Optimization Theory*, Cambridge University Press.