Endogenous Growth Theory: One-sector Models of Endogenous Growth

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This lecture note is mainly based on Ch. 4 of Barro and Sala–i–Martin (2004). Ch. 11 of Acemoglu (2009) also provides excellent explanations of one-sector endogenous growth models.

Contents

1 Introduction

- In the mid-1980s, it became clear that the standard Ramsey–Cass–Koopmans model was theoretically unsatisfactory as a tool to explore the determinants of long-run growth.

  ($\because$) We have already seen that the per-capita growth rate in the steady state equals

  - zero (in the baseline model), or
  - the rate of technological progress, (in the model of exogenous technological progress).

  This is due to the property that the marginal productivity of capital diminishes as it is accumulated over time.

  $\rightarrow$ It became a priority to go beyond the treatment of technological progress as exogenous.

- (Conceptual Difficulties of Endogenous Technological Progress)

  - Consider a firm i, of which production is according to

    $$Y_{it} = F(K_{it}, A_{it}L_{it}),$$

    where

    * Y_i, K_i and L_i: output, capital, labor of firm i,
    * A_i: the technology of this firm.

  - It is assumed that F is homogenous of degree one:

    $$F = \frac{\partial F}{\partial K_i}K_i + \frac{\partial F}{\partial L_i}L_i \left(= \frac{\partial F}{\partial K_i}K_i + \frac{\partial F}{\partial A_i L_i}A_i L_i\right)$$

  - Then, the perfect competition results in zero profit:

    $$F = RK_i + wL_i$$

    $\rightarrow$ There is no more resources to improve A_i.

- Then, in order to theoretically obtain the sustainable growth without relying on the exogenous technological progress, we need to

  1. assume that the technology is non-excludable and non-rival, or
  2. broaden the concept of capital to include human components, or
  3. introduce the framework of imperfect competition which motivates the firms to engage in R&D activities.

- Influential Papers and Books:
  - Paul Romer's seminal paper (Romer, 1986).
    → Learning by doing and knowledge spillovers (or – externalities) induces a technological change.
    (∗) The work of Kenneth Arrow formed the basis for this research (Arrow, 1962).

  - Human capital accumulation, by Robert Lucas (Lucas, 1988).
    (∗) The work of Hirofumi Uzawa formed the basis for this research (Uzawa, 1965).

  - Productive public service, by Robert J. Barro (1990).

  - Endogenous technological change by R&D, innovation, and imperfect competition
    → Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992).
- This note covers
  1. AK model.
  2. Romer's (1986) model.

2 AK Model

- AK model is a simple prototype of endogenous growth, based on the theory of "capital in a broader sense (to include human capital)."
- Since Rebelo (1991), the AK model has been extensively used to analyze the effects of various policies on the growth rate.

  (∗) Barro and Sala-i-Martin (2004, Ch. 1) state that: " (w)e think that the first economist to use a production function of the AK type was von Neumann (1937)."
- Properties of dynamics, especially, the so-called "no transition" result is satisfied in many other types of endogenous growth. So we first analyze this model.

2.1 Behaviors of Firms and Households

Households

- Utility maximization problem of a representative household:

$$\begin{aligned}
\max \quad & U_0 = \int_0^\infty e^{-(\rho-n)t} u(c_t) dt \\
\text{s.t.} \quad & \dot{a}_t = (r_t - n)a_t + w_t - c_t \quad \text{(flow budget constraint)}, \\
& a_0 \text{ given} \quad \text{(initial assets)} \\
& \lim_{t\to\infty} a_t \exp\left(-\int_0^t (r_s - n) ds\right) \geq 0 \quad \text{(the No-Poinzi game condition)}.
\end{aligned}$$

- As in the Ramsey model, $\rho > n$ is assumed.
- Furthermore, to obtain the balanced growth path (BGP), we need to assume

Assumption 1. *The instantaneous utility function u is specified as $u(c) = \dfrac{c^{1-\theta} - 1}{1-\theta}$ for $\theta > 0, \theta \neq 1$, and $u(c) = \ln c$ for $\theta = 1$.*

- Current-value Hamiltonian is
$$H(a_t, c_t, \mu_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} + \mu_t\big[(r_t - n)a_t + w_t - c_t\big].$$
- Necessary and Sufficient conditions for utility maximization:
$$c_t^{-\theta} = \mu_t, \quad \dot{\mu}_t = (\rho - r_t)\mu_t, \quad \lim_{t\to\infty} \mu_t a_t \exp[-(\rho - n)t] = 0,$$
and the flow budget constraint.
- These conditions are reduced to
$$\dot{a}_t = (r_t - n)a_t + w_t - c_t, \tag{1}$$
$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho), \tag{2}$$
$$\lim_{t\to\infty} \left[a_t \exp\left(-\int_0^t (r_s - n)ds\right)\right] = 0. \tag{3}$$

Firms

- The only departure from the standard Ramsey model is that all firms have the linear production function.
- Then, aggregate output is $Y_t = AK_t$, which yields
$$y_t = f(k_t) = Ak_t.$$
- Two important features:
  - The marginal product of capital is NOT diminishing: $f''(k) = 0$.
  - The Inada condition is violated: $f'(k) = A$ for all $k \geq 0$.
- Conditions for profit maximization yields $A = R_t \forall t$.

  Note: $w_t = 0$.

2.2 Equilibrium

Definition of Competitive Equilibrium Path

- Asset market equilibrium: $a_t = k_t$.

$$r_t = A - \delta. \tag{4}$$

- Substituting this, and firms' optimality conditions: (4) and $w_t = 0$ into (1), (2) and (3):

$$\dot{k}_t = (A - \delta - n)k_t - c_t, \tag{5}$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(A - \delta - \rho), \tag{6}$$

$$\lim_{t\to\infty} k_t \exp[-(A - \delta - n)t] = 0. \tag{7}$$

(5)–(7) jointly constitute the dynamical system of this economy.

Definition 1. *Given $k_0 > 0$, the path of (k_t, c_t) which satisfies (5)–(7) is the equilibrium path.*

- From (6), the growth rate of consumption is constant from the initial time. To ensure positive growth, we assume that

Assumption 2 (Positive Growth)**.** $A - \delta - \rho > 0$.

- At the same time, U_0, must be finite under (6).

Assumption 3. $\left| \int_0^\infty e^{-(\rho-n)t} u(c_t) dt \right| < \infty$.

Lemma 1. *Let $\varphi \equiv \frac{\theta - 1}{\theta}(A - \delta) + \frac{\rho}{\theta} - n$. Assumption 3 is satisfied if and only if $\varphi > 0$.*

Proof. Exercise. □

- Both of Assumptions 2 and 3 are satisfied if and only if

$$A - \delta > \rho > (1 - \theta)(A - \delta) + \theta n. \tag{8}$$

Unique Existence of the Balanced Growth Path

- As already explained, if the equilibrium path converges to the balanced growth path (BGP), all per capita variables grow at the same rate.
- Let us introduce the following new variable:

$$z_t \equiv \frac{c_t}{k_t}.$$

Note: From the definition of z_t,

  - when z_t becomes constant, k_t grows at the same rate as c_t.
  - $z_0 \equiv c_0/k_0$ is endogenous, since c_0 is endogenous.

- From (5) and (6), we obtain

$$\begin{aligned} \dot{z}_t &\equiv z_t(\dot{c}_t/c_t - \dot{k}_t/k_t) \\ &= z_t \left[\underbrace{\frac{1}{\theta}(A-\delta-\rho)}_{=\dot{c}_t/c_t} - \underbrace{(A-\delta-n-z_t)}_{=\dot{k}_t/k_t} \right] \\ &= z_t\left(z_t - \varphi\right), \end{aligned} \quad (9)$$

where φ is defined in Lemma 1

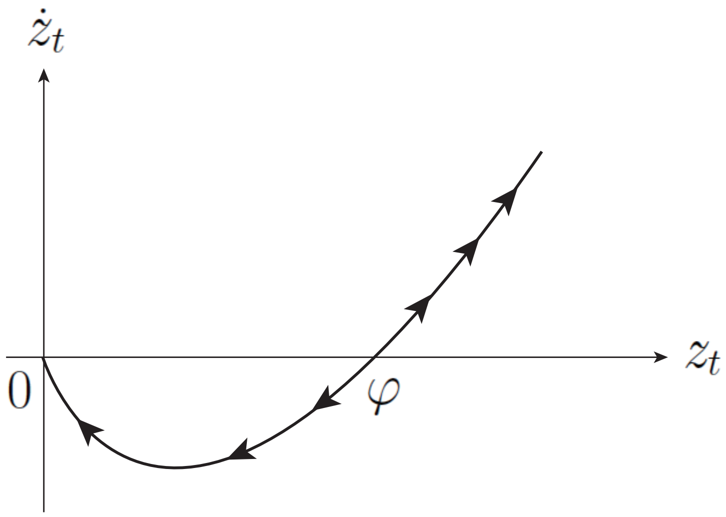

Figure 1: Dynamics of z_t

- Figure 1 depicts the phase diagram on the $(z_t,\, \dot{z}_t)$ plane.
- There are the following three candidates for z_0:

1. $z_0 = \varphi$: in this case, $z_t = \varphi$ for all $t \in [0, \infty)$.
   $\rightarrow \dot{k}_t/k_t = \dot{c}_t/c_t$ for all $t \in [0, \infty)$.
   $\rightarrow$ that is, the economy arrives at the BGP from the initial date.

   Check of the TVC: Substituting $k_t = k_0 \exp[(1/\theta)(A - \delta - \rho)t]$ into the LHS of (7),

$$\lim_{t \to \infty} k_t \exp[-(A - \delta - n)t] = \lim_{t \to \infty} k_0 \exp(-\varphi t).$$

   Since $\varphi > 0$ is assumed, this converges to 0.

2. $z_0 < \varphi$: from Figure 1, z_t decreases over time and eventually converges to 0.
   $\rightarrow$ Such a path violates the TVC.

3. $z_0 > \varphi$: from Figure 1, z_t diverges.
   $\rightarrow$ On such a path, k_t reaches 0 in finite time.

Proposition 1. *In the competitive equilibrium path, $z_t = \varphi$ ($c_t = \varphi k_t$) for all $t \in [0, \infty)$. That is, there is no transition and the economy reaches the BGP at the initial time.*

3 A Model with Learning-by-Doing and Knowledge Spillovers

- The key to sustainable growth in the AK model is the absence of diminishing returns to K.
- However, there are some shortcomings. A major one is the share of capital in national income, say, RK/Y is equal to one in this model. However, in reality, that is about 1/3.
- Paul Romer, in his influential paper in 1986, has constructed a model of endogenous growth in which knowledge spillovers play a central role.
- Consider an economy without any population growth:

Assumption 4. *There is no population growth: $L_t = L \forall t \in [0, \infty)$.*

- The household side is essentially same as the standard Ramsey model except $n = 0$.

3.1 Technology

- The production side of the economy consists of a set $[0,1]$ of firms.
- The production function of firm $i \in [0,1]$ is

$$Y_{it} = F(K_{it}, A_{it}L_{it}), \tag{10}$$

where

  - Y_i, K_i and L_i: output, capital, labor of firm i,
  - A_i: the index of knowledge available to the firm.

- Two important assumptions about productivity growth in Romer (1986) (and Arrow (1962)):

  1. Learning by doing works through each firm's net investment.
     $\rightarrow$ An increase in K_i leads to a parallel increase in the stock of knowledge.
  2. Each firm's knowledge stock is a public good (i.e., a good of non-rival and non-excludable properties).
     $\rightarrow$ $\dot{A}_{it}$ corresponds to $\dot{K}_t$, where $K_t \equiv \int_0^1 K_{it} di$.
     $\rightarrow$ $A_{it} = b_1 + b_2 K_t$. Hereafter $b_1 = 0$ and $b_2 = 1$.

- We combine the assumptions of learning by doing and knowledge spillovers.

$$Y_{it} = F(K_{it}, K_t L_{it}). \tag{11}$$

3.2 Firms' Behavior

- Key feature: K_t is taken as given for each firm, but endogenously determined for the economy as a whole (Marshallian externalities).
- Firm i's profit maximization:

$$\max_{K_{it}, L_{it}} F(K_{it}, K_t L_{it}) - R_t K_{it} - w_t L_{it}$$

- Conditions for profit maximization for firm i:

$$R_t = \frac{\partial F(K_{it}, K_t L_{it})}{\partial K_{it}}, \quad w_t = K_t \frac{\partial F(K_{it}, K_t L_{it})}{\partial (K_t L_{it})}. \tag{12}$$

- From (12), all firms make the same choice:

$$K_{it} = K_t, \quad L_{it} = L_t. \tag{13}$$

<u>Example:</u> If we specify F as a Cobb-Douglas form: $Y_{it} = K_{it}^{\alpha}(K_t L_{it})^{1-\alpha}$, (12) is simply given by

$$R_t = \alpha \left(\frac{K_{it}}{K_t L_{it}} \right)^{-(1-\alpha)}, \quad w_t = (1-\alpha) \left(\frac{K_{it}}{K_t L_{it}} \right)^{\alpha}.$$

3.3 Equilibrium

- Since F is homogenous of degree 1, $F(K_i, KL_i) = F(K_i/K, L_i)K$. From (13),

$$F(K, KL) = f(L)K, \text{where } f(L) = F(1, L). \tag{14}$$

  Then, by definition, $f'(L) = \dfrac{\partial F(1, L)}{\partial L}$.

- Since the partial derivative of F is homogenous of degree zero,

$$f'(L) = \frac{\partial F(1, L)}{\partial L} = \frac{\partial F(x, xL)}{\partial (xL)} \forall x \geq 0.$$

  From this property of (12),

$$w_t = f'(L)K_t. \tag{15}$$

- Then, using (12) and (15), the interest rate r_t is given by

$$\begin{aligned} r_t &= R_t - \delta \\ &= \frac{\partial F(K_{it}, K_t L_{it})}{\partial K_{it}} - \delta \\ &= f(L) - Lf'(L) - \delta. \end{aligned} \tag{16}$$

- Substituting (15), (16) and the asset market equilibrium $a_t = k_t$ into the household budget constraint (1),

$$\begin{aligned} \dot{k}_t &= (R_t - \delta)k_t + w_t - c_t \\ &= (f(L) - \delta)k_t - c_t. \end{aligned} \tag{17}$$

- On the other hand, substituting (16) into the Euler equation (2), the dynamics of per capita consumption is given by

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}(R_t - \delta - \rho) \\ &= \gamma^*, \end{aligned} \tag{18}$$

  where

$$\gamma^* \equiv \frac{1}{\theta}\left(f(L) - Lf'(L) - \delta - \rho\right).$$

- Finally, the TVC in this model is given by

$$\lim_{t\to\infty} k_t \exp\left[-(f(L) - Lf'(L) - \delta)t\right] = 0. \tag{19}$$

  (17)-(19) jointly constitute the dynamical system.

Definition 2. *Given $k_0 > 0$, the path of $(k_t,\, c_t)$ which satisfies (17)–(19) is the equilibrium path of the model with learning by doing and knowledge spillovers.*

- Since $f(L) = F(1, L)$, $f(L)$ is strictly concave with respect to L. Then, $f(L) - Lf'(L) > 0$ holds for all L.

  Example: If we specify F as the aforementioned Cobb-Douglas form, $f(L) = L^{1-\alpha}$.

$$f(L) - Lf'(L) = \alpha L^{1-\alpha} > 0.$$

- The condition (8) is replaced by

$$f(L) - Lf'(L) - \delta > \rho > (1-\theta)(f(L) - Lf'(L)). \tag{20}$$

  That is, as long as (20) is satisfied, the positive growth and the boundedness of U_0 are guaranteed.

Proposition 2. *Given $k_0 > 0$. there exists a unique equilibrium path where capital, consumption, and output grow at the rate of $\gamma^* \equiv (1/\theta)(f(L) - Lf'(L) - \delta - \rho)$ from the initial time:*

$$k_t = k_0 \exp\left(\gamma^* t\right), \quad c_t = \tilde{\varphi} k_t, \quad y_t = f(L) k_t,$$

where

$$\tilde{\varphi} = \frac{1}{\theta}\left[(\theta - 1)(f(L) - \delta) + Lf'(L) + \rho\right].$$

Proof. Exercise. □

- The growth rate depends on
  - the level of technology (f),
  - the willingness to save (ρ), and
  - the population (L).
- The growth effect of the population is discussed in Section 3.5.
- Introducing public policies into this model enables us to examine how the growth rate is affected by such policies.

3.4 Suboptimality of Equilibrium Allocation

- Not only from the model by itself, the importance of Paul Romer's paper stems from its emphasis on the spillovers of knowledge and ideas.
- Although such a nonrival nature makes a sustainable growth possible, at the same time it makes the equilibrium allocation inferior to the first-best allocation.
- Consider the following problem by a social planner:

$$\begin{aligned} \max \quad & U_0 = \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta} - 1}{1-\theta} dt \\ \text{s.t.} \quad & \dot{k}_t = f(L)k_t - \delta k_t - c_t, \\ & k_0 > 0 \text{ given.} \end{aligned}$$

- The current-value Hamiltonian is

$$\hat{H}(k_t, c_t, \mu_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} + \mu_t[f(L)k_t - \delta k_t - c_t].$$

- The necessary and sufficient conditions for the socially optimal path are

$$\frac{\partial \hat{H}}{\partial c} = 0 \ \Leftrightarrow c_t^{-\theta} = \mu_t, \tag{21}$$

$$\dot{\mu}_t = \rho \mu_t - \frac{\partial \hat{H}}{\partial a} \ \Leftrightarrow \dot{\mu}_t = [\rho - (f(L) - \delta)]\mu_t, \tag{22}$$

$$\lim_{t\to\infty} \mu_t k_t \exp(-\rho t) = 0, \tag{23}$$

and the resource constraint.

- Using these equations, the socially optimal path is characterized by

$$\dot{k}_t = f(L)k_t - \delta k_t - c_t, \tag{24}$$

$$\frac{\dot{c}_t}{c_t} = \gamma^S, \tag{25}$$

$$\lim_{t\to\infty} k_t \exp[-(f(L) - \delta)t] = 0, \tag{26}$$

where

$$\gamma^S \equiv \frac{1}{\theta}(f(L) - \delta - \rho).$$

- We can easily find (24)–(26) is equivalent to (5)–(7) if we replace $f(L)$ by A.

  → The qualitative feature of the optimal path is essentially the same as the baseline AK model in Section 2.

- The optimal path settles to the BGP from the beginning, and the growth rate is given by

$$\dot{c}_t/c_t = \dot{k}_t/k_t = \dot{y}_t/y_t = \gamma^S. \tag{27}$$

  This rate is always greater than the growth rate under the laissez-faire economy, since $f(L) > f(L) - Lf'(L)$.

Proposition 3. *In the endogenous growth model of learning by doing and knowledge spillovers, the equilibrium allocation is inferior to the first best allocation, and grows at a slower rate than that.*

3.5 Scale Effects

- The growth rate in this model:
  - $\gamma^* \equiv (1/\theta)(f(L) - Lf'(L) - \delta - \rho)$ on the market equilibrium path,
  - $\gamma^S \equiv (1/\theta)(f(L) - \delta - \rho)$ on the socially optimal path.

- This means that an expansion of the aggregate labor force L raises the per capita growth rate, which induces the following two problems:

  1. There may be no BGP if the labor force grows over time, i.e., if $\dot{L}_t/L_t > 0$. To see why, note that the dynamical system in this case is

$$\dot{z}_t/z_t = \frac{1}{\theta}(f(L_t) - L_t f'(L_t) - \delta - \rho) - f(L_t) + \delta + n + z_t, \tag{28}$$

$$\dot{L}_t/L_t = n. \tag{29}$$

  2. If we can identify L with the aggregate labor force of a country, the prediction is that countries with more workers tend to grow faster *even in* per capita terms.
     → Empirical studies do not sufficiently supports this prediction: the growth rate of per capita GDP bears little relation to the country's population size.

References

[1] Acemoglu, D. (2009) *Introduction to Modern Economic Growth*, Princeton University Press.

[2] Aghion, P. and P. Howitt (1992) "A Model of Growth through Creative Destruction." *Econometrica* 60, pp. 323–351.

[3] Arrow, K. J. (1962) "The Economic Implications of Learning by Doing." *Review of Economic Studies* 29, pp. 155–173.

[4] Barro, R. J. (1990) "Government Spending in a Simple Model of Endogenous Growth." *Journal of Political Economy* 98, S103–S125.

[5] Barro, R. J. and X. Sala-i-Martin (2004) *Economic Growth*, Second Edition, Cambridge, MIT Press.

[6] Grossman, G., and E. Helpman (1991) *Innovation and Growth in the Global Economy*, Cambridge MA, MIT Press.

[7] Lucas, R. E. (1988) "On the Mechanics of Economic Development." *Journal of Monetary Economics* 22, pp. 3–42.

[8] Rebelo, S. (1991). "Long Run Policy Analysis and Long Run Growth." *Journal of Political Economy* 99, pp. 500–521.

[9] Romer, P. M. (1986) "Increasing Returns and Long-Run Growth." *Journal of Political Economy* 94, pp. 1002–1037.

[10] Uzawa, H. (1965) "Optimum Technical Change in an Aggregative Model of Economic Growth. *International Economic Review* 6, pp. 18-31.

[11] Von Neumann, J. (1945) " A Model of General Equilibrium." *Review of Economic Studies* 13, pp. 1–9.