Ramsey–Cass–Koopmans Model (2): Some Extensions

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This lecture note is mainly based on Ch. 8 of Acemoglu (2009). Ch. 2 of Blanchard and Fischer (1989) and Chs. 2–3 of Barro and Sala–i–Martin (2004) also provide excellent explanations of the Ramsey–Cass–Koopmans model.

Contents

1 The Role of Public Policies

We now introduce the government into the model. Topics to be covered are

1. Effects of government spending
2. Effects of debt financing
3. Effects of taxations

1.1 Government Spending

Government Spending under Balanced Budget

- Suppose that the government consumes G_t units of the final good.

  $\rightarrow$ in per capita terms, $g_t = G_t/L_t$.

- It levies lump-sum taxes T_t to finance the expenditure. Therefore the government's budget constraint is
$$T_t/L_t = g_t. \tag{1}$$

- We assume the path of g_t is exogenously given. Then the above equation determines the path of T_t.

Equilibrium

- The household's flow budget constraint now becomes
$$\dot{a}_t = (r_t - n)a_t + w_t - c_t - T_t/L_t, \tag{2}$$

- The household takes the path of T_t as given. Therefore the Euler equation does not change.

  $\downarrow$

- Then, it is found that the dynamics of c_t is essentially same as that in the economy without the government:
$$-\frac{c_t u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} = f'(k_t) - \delta - \rho.$$

- On the other hand, the dynamics of k_t now becomes
$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t - g_t.$$

- Figure 1: The Phase diagram when g_t is exogenously constant over time. Hereafter we assume that g_t is exogenously constant over time: $g_t = g$.

- What happens if g increases?

  $\rightarrow$ In steady state government spending completely crowds out private consumption, *but has no effect on the capital stock*

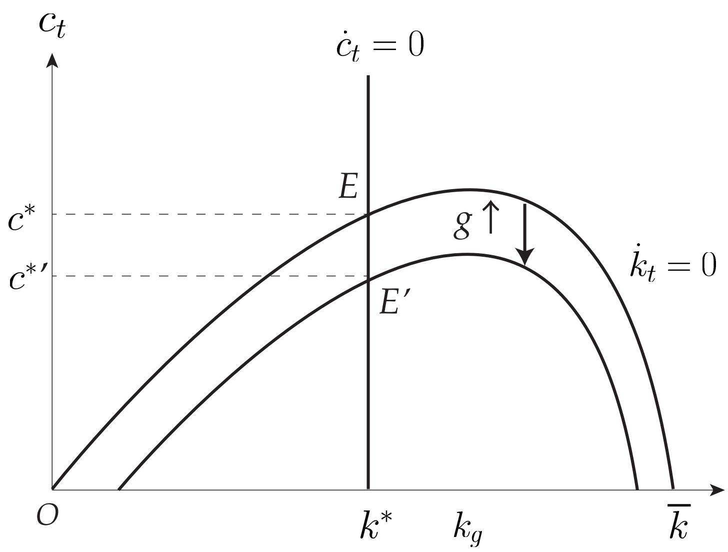

Figure 1: Effects of government spending

1.2 Debt Financing and the Ricardian Neutrality

Government's Intertemporal Budget Constraint under Debt Financing

- Now relax the balanced-budget assumption (1). The government is now allowed to borrow, instead of financing itself only through lump-sum taxes.
- Let $B_t \geq 0$ denote stock of government debt at date t. The government's budget constraint is now given by

  $$\underbrace{T_t + \dot{B}_t}_{\text{Revenue}} = \underbrace{r_t B_t + G_t}_{\text{Expenditure}},$$

  or equivalently,

  $$\dot{B}_t = \underbrace{r_t B_t + \underbrace{G_t - T_t}_{\text{Primary deficit}}}_{\text{Budget deficit}}.$$

- Integrating the above equation from zero to infinity,

  $$B_0 = \int_0^\infty (T_t - G_t) \exp\left(-\int_0^t r_s ds\right) dt + \lim_{t\to\infty} B_t \exp\left(-\int_0^t r_s ds\right).$$

- The No-Poinzi Game condition which prohibits the government to default is

  $$\lim_{t\to\infty} B_t \exp\left(-\int_0^t r_s ds\right) = 0,$$

  which leads the following intertemporal budget constraint of the government:

  $$B_0 = \int_0^\infty (T_t - G_t) \exp\left(-\int_0^t r_s ds\right) dt. \tag{3}$$

Ricardian Neutrality

- The asset market equilibrium is now given by

$$A_t = K_t + B_t.$$

- Then, the households' aggregate intertemporal budget constraint is

$$\int_0^\infty C_t \exp\left(-\int_0^t r_s ds\right) dt = K_0 + B_0 + \int_0^\infty (w_t L_t - T_t) \exp\left(-\int_0^t r_s ds\right) dt. \tag{4}$$

- Then, substituting (3) into (4) yields

$$\int_0^\infty C_t \exp\left(-\int_0^t r_s ds\right) dt = K_0 + \int_0^\infty (w_t L_t - G_t) \exp\left(-\int_0^t r_s ds\right) dt. \tag{5}$$

  Notice that

  1. Neither taxes T_t nor the debt B_t appears in the budget constraint,
  2. Only government spending G_t matters.

  This result is summarized as follows:

Proposition 1. *For a given path of G_t, financing it through distortionless taxation and budget deficit are indifferent.*

In other words, the method of finance, whether distortionless taxation or budget deficit has no effect on equilibrium allocation. This property is called the *Ricardian neutrality* or *Ricardian equivalence.*

1.3 Taxations

Effects of Consumption Tax

- We now consider the following household' budget constraint:

$$\dot{a}_t = (r_t - n)a_t + w_t - (1 + \tau_t^c)c_t - T_t/L_t, \tag{6}$$

  where $\tau_t^c \geq 0$ is the rate of consumption tax.

- The current–value Hamiltonian:

$$H(a_t, c_t, \mu_t) = u(c_t) + \mu_t[(r_t - n)a_t + w_t - (1 + \tau_t^c)c_t - T_t/L_t]. \tag{7}$$

- The conditions for utility maximization are

$$\partial H/\partial c_t = 0 \Leftrightarrow u'(c_t) = \mu_t(1+\tau_t^c), \tag{8}$$
$$\partial H/\partial a_t = \dot{\mu}_t - (\rho - n)\mu_t \Leftrightarrow \dot{\mu}_t/\mu_t = \rho - r_t, \tag{9}$$
$$\lim_{t\to\infty} \mu_t a_t e^{-(\rho-n)t} = 0. \tag{10}$$

- If τ_t^c is constant over time, $\dot{\tau}_t^c = 0$. Then, from (8) and (9),

$$\begin{aligned} -\frac{c_t u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} &= -\dot{\mu}_t/\mu_t \\ &= r_t - \rho. \end{aligned}$$

Thus, the Euler equation does not change.

- We can easily obtain the same result even though we extend the *many goods-economy*. Let c_{jt} denote consumption of good $j = 1, 2, \ldots, J$. The representative household's utility problem is

$$\begin{aligned} \max \quad & \int_0^\infty e^{-(\rho-n)t} u(c_{1t}, c_{2t}, \ldots, c_{Jt}) dt \\ \text{s.t.} \quad & \dot{a}_t = (r_t - n)a_t + w_t - (1+\tau_t^c)\sum_{j-1}^{J} p_{jt}c_{jt} - T_t/L_t, \\ & a_0 \text{ and NPG}, \end{aligned}$$

where p_{jt} is the price of good j. Without any loss of generality, p_{1t} is normalized to 1.

($*$) Due to the Walras' law, we must normalize the price of one good to unity.

- Now (8) is now rewritten as

$$\partial u(\cdot)/\partial c_{jt} = p_{jt}\mu_t(1+\tau_t^c), \;\; j = 1, 2, \ldots, J.$$

This leads the following well-known formula, which implies the marginal rate of substitution is equal to the relative price:

$$\frac{\partial u(\cdot)/\partial c_{jt}}{\partial u(\cdot)/\partial c_{1t}} = p_{jt}, \;\; j = 2, 3, \ldots, J. \tag{11}$$

Thus, the tax rate disappears.

Proposition 2. *An increase in the consumption tax rate has no effect on the equilibrium allocation if*

(i) such an increase applies for all goods, and

(ii) the rate of taxation remains to be flat ($d\tau_t$ is constant over time).

Caution 1: Possibility of Static Distortion If we introduce a "labor-leisure choice" by the household, consumption tax can distort her decision making about her leisure.

$\rightarrow$ Exercise.

Caution 2: Possibility of Dynamic Distortion If the tax rate on consumption is *anticipated* to increase in the future, the households would want to consume more now and less in the future.

Effects of Capital Income Tax

- We now consider the following household' budget constraint

$$\dot{a}_t = [(1-\tau_t^a)r_t - n]a_t + w_t - c_t - T_t/L_t, \tag{12}$$

where $\tau_t^a \in [0, 1)$ is the capital income tax rate.

- The current–value Hamiltonian:

$$H(a_t, c_t, \mu_t) = u(c_t) + \mu_t\big\{[(1-\tau_t^a)r_t - n]a_t + w_t - c_t - T_t/L_t\big\}. \tag{13}$$

- The conditions for utility maximization are

$$\partial H/\partial c_t = 0 \Leftrightarrow u'(c_t) = \mu_t, \tag{14}$$

$$\partial H/\partial a_t = \dot{\mu}_t - (\rho - n)\mu_t \Leftrightarrow \dot{\mu}_t/\mu_t = \rho - (1-\tau_t^a)r_t, \tag{15}$$

$$\lim_{t\to\infty} \mu_t a_t e^{-[(1-\tau_t^a)r_t - n]t} = 0. \tag{16}$$

- Then, from (14) and (15),

$$-\frac{c_t u''(c_t)}{u'(c_t)}\frac{\dot{c}_t}{c_t} = (1-\tau_t^a)r_t - \rho.$$

- To focus on the effects of taxation, we assume that the government does not issue the public bond and $g_t = 0$. Namely, the government's budget constraint is

$$\tau^a r_t A_t + T_t = 0. \tag{17}$$

This means $T_t < 0$. The capital income tax revenue is used for the redistribution to the households.

- Dynamic system under capital income taxation:

$$\dot{k}_t = f(k_t) - (n+\delta)k_t - c_t,$$

$$-\frac{c_t u''(c_t)}{c_t}\frac{\dot{c}_t}{c_t} = (1-\tau^a)f'(k_t) - \delta - \rho,$$

$$\lim_{t\to\infty} k_t \exp\left(-\int_0^t (f'(k_s) - (n+\delta))ds\right) = 0.$$

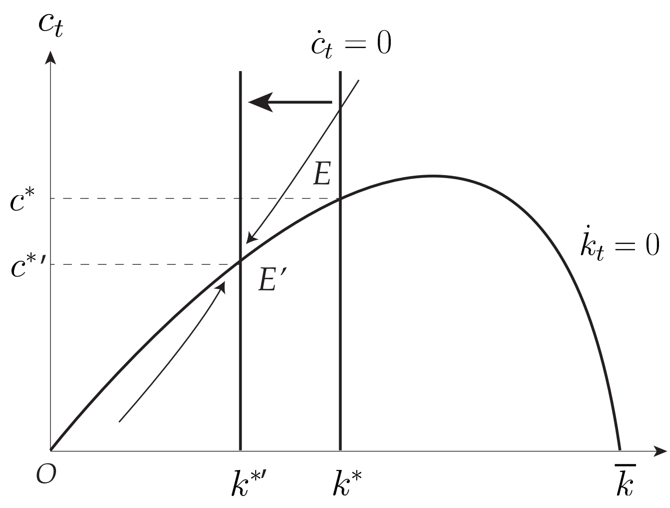

Figure 2: Effets of capital income taxation

- Figure 2 shows how the capital income taxation affects the economy. The steady state moves from E to E':

  1. the steady state capital stock falls from k^* to $k^{*'}$.
  2. the steady state consumption also falls from c^* to $c^{*'}$.

  However, note that during the transition, consumption increases temporarily.

2 Exogenous Technological Progress

- In the baseline model, the equilibrium path of (k_t, c_t) converges to the steady state (k^*, c^*).

  $\rightarrow$ The growth rate of all variables in per capita terms eventually becomes zero.

- To make the model more realistic, now we introduce the technological progress into the baseline model.

2.1 Labor–augmenting Technological Progress

- We extend the production function to

$$Y_t = F(K_t, Z_t L_t),$$

  Even if K_t or L_t does not change, Y_t increases if Z_t increases. Hereafter we interpret Z_t as the level of technology.

- We introduce changes in Z_t to capture improvements in the technological know-how of the economy.

$$\dot{Z}_t / Z_t = \gamma > 0, \tag{18}$$

or equivalently

$$Z_t = Z_0 \exp(\gamma t). \tag{19}$$

The technological progress such as (18) or (19) is called the *Labor–augmenting Technological Progress.*

- Define the following new variables:

$$\widetilde{y}_t \equiv \frac{Y_t}{Z_t L_t}, \quad \widetilde{k}_t \equiv \frac{K_t}{Z_t L_t}$$

- We continue to assume F is constant returns to scale, and define the function f as

$$f(\widetilde{k}) \equiv F(\widetilde{k}, 1)$$

- The representative firm's profit maximization problem:

$$\max_{\widetilde{k}_t, L_t} \quad [f(\widetilde{k}_t) - R_t \widetilde{k}_t - (w_t/Z_t)] Z_t L_t$$

The first–order–conditions are

$$R_t = f'(\widetilde{k}_t), \tag{20}$$

$$w_t = [f(\widetilde{k}_t) - \widetilde{k}_t f'(\widetilde{k}_t)] Z_t, \tag{21}$$

which leads

$$R_t \widetilde{k}_t + w_t/Z_t = \widetilde{y}_t. \tag{22}$$

2.2 Household's Maximization

- The conditions for utility maximization are the same as those in the baseline model:

$$\dot{a}_t = (r_t - n) a_t + w_t - c_t, \tag{23}$$

$$-\frac{c_t u''(c_t)}{u'(c_t)} \frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{24}$$

$$\lim_{t \to \infty} a_t \exp\left(- \int_0^t (r_s - n) ds \right) = 0. \tag{25}$$

2.3 Balanced Growth Path

- Asset market equilibrium does not change from the baseline model.

$$a_t = k_t, \tag{26}$$

which implies

$$r_t = R_t - \delta. \tag{27}$$

- Substituting (26) and (27) into (23) and applying the firm's zero profit condition (22) into the resulting equation, we have

$$\dot{k}_t = f(\widetilde{k}_t)Z_t - (n+\delta)k_t - c_t.$$

- Then, the dynamics of $\widetilde{k}_t$ is given by

$$\begin{aligned} & \dot{\widetilde{k}}_t/\widetilde{k}_t = \dot{k}_t/k_t - \gamma \\ \to \quad & \dot{\widetilde{k}}_t = f(\widetilde{k}_t) - (n+\delta+\gamma)k_t - \widetilde{c}_t, \end{aligned} \tag{28}$$

where

$$\widetilde{c}_t \equiv \frac{C_t}{Z_t L_t}.$$

- On the other hand, substituting (20) and (27) into (24), the dynamics of $\widetilde{c}_t$ is given by

$$\begin{aligned} \frac{\dot{\widetilde{c}}_t}{\widetilde{c}_t} &= \frac{\dot{c}_t}{c_t} - \gamma \\ &= \left(-\frac{c_t u''(c_t)}{u'(c_t)}\right)^{-1} (f'(\widetilde{k}_t) - \delta - \rho) - \gamma \end{aligned}$$

Thus, $-cu''/u'$ must be constant.

Assumption 1. *$u(c) = \dfrac{c^{1-\theta}-1}{1-\theta}$, where $\theta > 0$ and $\theta \neq 1$.*

When $\theta = 1$, u is specified as $u(c) = \ln c$.

- The dynamics of $\hat{c}_t$ is eventually given by

$$\frac{\dot{\widetilde{c}}_t}{\widetilde{c}_t} = (1/\theta)(f'(\widetilde{k}_t) - \delta - \rho - \theta\gamma). \tag{29}$$

- Finally, the TVC (25) is reduced to

$$\lim_{t\to\infty} \widetilde{k}_t \exp\left(-\int_0^t (f'(\widetilde{k}_s) - n - \gamma)ds\right) = 0. \tag{30}$$

(28)–(30) jointly constitute the dynamic system.

- Since in steady state $\widetilde{k}_t$ and $\widetilde{c}_t$ must remain constant, from (28) and (29) we have

$$f'(\widetilde{k}^*) = \rho + \delta + \theta\gamma, \tag{31}$$

and

$$\widetilde{c}^* = f'(\widetilde{k}^*) - (n+\delta+\gamma)\widetilde{k}^*. \tag{32}$$

- We can show the unique existence of $(\widetilde{k}^*, \widetilde{c}^*)$ which solves (31) and (32) in a way similar to the model without technological progress.

- The only additional condition in this case is that because there is growth, we have to make sure that the TVC is in fact satisfied.

  → Substituting (31) into (30), we have

$$\lim_{t\to\infty} \widetilde{k}_t \exp\{-[\rho - n - (1-\theta)\gamma]t\} = 0.$$

  which can only hold if the following assumption is satisfied:

Assumption 2. $\rho - n > (1-\theta)\gamma$.

- In the steady state, $\widetilde{k}_t$ and $\widetilde{c}_t$ are constant over time.

- From these definitions,

$$\dot{k}_t/k_t = \dot{c}_t/c_t = \gamma. \tag{33}$$

  Furthermore, since $y_t = \widetilde{y}_t Z_t = f(\widetilde{k}_t)Z_t$, the growth rate of per capita GDP is γ in the long run.

  → In the steady state, all per capita variables grow at the rate of $\gamma > 0$.

  → In this model, the steady state is called the *Balanced Growth Path* (均斉成長経路).

Proposition 3 (Balanced Growth Path)**.** *In steady state all per capita variable grow at the constant rate of technological progress, $\gamma > 0$.*

3 Summary

- In the baseline Ramsey model, the government spending crowds out private consumption.

- Consumption tax is distortionless as long as the same rate applies for all goods and it is constant over time.

- Capital income taxation harms the households' savings, thereby capital accumulation. In consequence, the capital stock in the steady state decreases if the tax rate becomes higher.

- By introducing the labor-augmenting technological change to the baseline model, all per capita variables (per capita GDP, capital, consumption...) become to grow at the same constant rate of technological progress in the long-run.

References

[1] Acemoglu, D. (2009) *Introduction to Modern Economic Growth*, Princeton University Press.

[2] Barro, R. J. and X. Sala-i-Martin (2004) *Economic Growth*, Second Edition, Cambridge, MIT Press.

[3] Blanchard, O. and S. Fischer (1989) *Lectures on Macroeconomics*, Cambridge, MIT Press.