An Introduction to Optimal Control Theory in Continuous Time

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Note: For more general explanations of optimal control problems in economics, see, for instance, Kamien and Schwartz (1981), Mathematical Appendix of Barro and Sala-i-Martin (2004) and Acemoglu (2009, Ch.7). This note is heavily influenced by theirs and borrows much from their insights.

1 Finite-horizon Optimal Control

The canonical continuous-time optimization problem is given by:

$$\begin{aligned} \max \quad & J = \int_0^T F(x(t), c(t), t)dt \\ \text{subject to} \quad & \dot{x}(t) \equiv \frac{dx(t)}{dt} = g(x(t), c(t)) \quad 0 \le t \le T, \\ & x(T) \ge 0, \quad x(0) \text{ given.} \end{aligned} \tag{P}$$

where

- J is the *objective functional*, while $F(\cdot)$ is the *one-period return function.*
- $c : [0, T] \to C \subseteq \mathbb{R}_+$ is called the *control variable.* It can be a vector. This kind of variables is immediately under the control of the decision maker.
- On the other hand, $x : [0, T] \to X \subseteq \mathbb{R}_+$, is called the *state variable.* It can be a vector. This kind of variables is determined only indirectly thorough

  $$\dot{x}(t) = g(x(t), c(t)),$$

  where the above equation is called the *transition equation.* A Solution to the problem (P) is called the *optimal path*, or the *optimal trajectory.*

1.1 Pontryagin's Maximum Principle

Assumptiom 1. *Both of F and g are continuously differentiable.*

Define the following Hamiltonian:

$$H(x(t), c(t), \lambda(t), t) \equiv F(x(t), c(t), \lambda(t), t) + \lambda(t)g(x(t), c(t)), \tag{1}$$

where $\lambda(t) \geq 0$ is called the *costate variable* or the *adjoint variable*. Note that since F and g are continuously differentiable, so is H.

To simplify notation, hereafter we let H_x, H_c and H_λ denote the partial derivatives of H with respect to x, c and λ, respectively:

$$H_x(x, c, \lambda, t) = \frac{\partial H(x, c, \lambda, t)}{\partial x},$$
$$H_c(x, c, \lambda, t) = \frac{\partial H(x, c, \lambda, t)}{\partial c},$$
$$H_\lambda(x, c, \lambda, t) = \frac{\partial H(x, c, \lambda, t)}{\partial \lambda},$$

Theorem 1 (**Simplified Maximum Principle).** *Consider the problem (P). Suppose that this problem has an interior continuous solution $c^*(t)$ and the corresponding interior continuous path of the state variable $x^*(t)$. Then, there exists $\lambda(t) \geq 0$ such that $x^*(t)$ and $c^*(t)$ satisfy the following conditions:*

$$H_c(x^*(t), c^*(t), \lambda(t), t) = 0 \quad \forall t \in [0, T], \tag{2}$$
$$\dot{\lambda}(t) = -H_x(x^*(t), c^*(t), \lambda(t), t) \quad \forall t \in [0, T], \tag{3}$$
$$\dot{x}(t) = H_\lambda(x^*(t), c^*(t), \lambda(t), t) \quad \forall t \in [0, T], \tag{4}$$

and the following terminal condition:

$$x^*(T) \geq 0,\ \lambda(T) \geq 0,\ \lambda(T)x^*(T) = 0. \tag{5}$$

Proof. Take an arbitrary continuous function $\gamma(t)$ and ε be a real number. Then, a variation of the function $c^*(t)$ is defined by

$$c(t, \varepsilon) = c^*(t) + \varepsilon\gamma(t).$$

We assume that ε is sufficiently small such that $c(t, \varepsilon)$ is feasible. By its definition,

$$c(t, 0) = c^*(t) \forall t \in [0, T].$$

Let us also define $x(t, \varepsilon)$ as the path of the state variable corresponding to the path of control variable $c(t, \varepsilon)$. We assume that also $x(t, \varepsilon)$ is feasible: i.e., $x(t, \varepsilon)$ satisfies

$$\dot{x}(t, \varepsilon) \left(\equiv \frac{dx(t, \varepsilon)}{dt} \right) = g(x(t, \varepsilon), c(t, \varepsilon)). \tag{6}$$

Since the initial state is historically given, $x(0, \varepsilon) = x(0)$ must hold. Then,

$$x(t, 0) = x^*(t) \forall t \in [0, T].$$

Now we can define

$$\mathcal{J}(\varepsilon) = \int_0^T F(x(t, \varepsilon), c(t, \varepsilon), t)dt. \tag{7}$$

Since (6) holds for all $t \in [0,T]$, we can verify that it follows that for any $\lambda(t)$,

$$\int_0^T \lambda(t)[g(x(t,\varepsilon), c(t,\varepsilon)) - \dot{x}(t,\varepsilon)] = 0. \tag{8}$$

Adding (8) to (7) yields:

$$\begin{aligned}\mathcal{J}(\varepsilon) &= \int_0^T \Big\{F(x,c,t) + \lambda(t)\big[g(x(t,\varepsilon), c(t,\varepsilon)) - \dot{x}(t,\varepsilon)\big]\Big\}dt \\ &= \int_0^T \Big\{H(x(t,\varepsilon), c(t,\varepsilon), \lambda(t), t) - \lambda(t)\dot{x}(t,\varepsilon)\Big\}dt. \end{aligned}\tag{9}$$

Integrating the term $\lambda(t)\dot{x}(t,\varepsilon)$ by parts, we obtain

$$\int_0^T \lambda(t)\dot{x}(t,\varepsilon)dt = \lambda(T)x(T,\varepsilon) - \lambda(0)x(0) - \int_0^T x(t,\varepsilon)\dot{\lambda}(t)dt = 0,$$

where we used the fact that $x(0,\varepsilon) = x(0)$ since the initial condition is historically given. Substituting this expression into (9) leads the following equation:

$$\mathcal{J}(\varepsilon) = \int_0^T \Big\{H(x(t,\varepsilon), c(t,\varepsilon), \lambda(t), t) + x(t,\varepsilon)\dot{\lambda}(t)\Big\}\, dt + \lambda(0)x(0) - \lambda(T)x(T,\varepsilon). \tag{10}$$

Now define $\mathcal{L}$ as follows:

$$\mathcal{L}(\varepsilon,\zeta) = \mathcal{J}(\varepsilon) + \zeta x(T,\varepsilon),$$

where ζ is the Kuhn-Tucker multiplier associated with the constraint on $x(T,\varepsilon) \geq 0$. Then, from the Kuhn-Tucker conditions, if $x^*(t)$ and $c^*(t)$ are optimal, the following conditions must be satisfied:

$$\frac{\partial \mathcal{L}(0,\zeta)}{\partial \varepsilon} = 0, \tag{11}$$

$$x(T,0) \geq 0,\ \zeta \geq 0,\ \zeta x(T,0) = 0. \tag{12}$$

From (10), (11) is rewritten as

$$\begin{aligned}\frac{\partial \mathcal{L}(0,\zeta)}{\partial \varepsilon} = 0 &\Leftrightarrow \mathcal{J}'(0) + \zeta\frac{\partial x(T,0)}{\partial \varepsilon} = 0 \\ &\Leftrightarrow \int_0^T \left\{H_c(x^*(t), c^*(t), \lambda(t), t) \times \gamma(t) + \Big[H_x(x^*(t), c^*(t), \lambda(t), t) + \dot{\lambda}(t)\Big]\frac{\partial x(t,0)}{\partial \varepsilon}\right\} dt \\ &- (\lambda(T) - \zeta)\frac{\partial x(T,0)}{\partial \varepsilon} = 0.\end{aligned}$$

If there would exist some $\gamma(t) \neq 0$ such that $\mathcal{J}'(0) > 0$, the objective J could be increased by deviating from the $(x^*(t), c^*(t))$. if this would be the case, this contradicts to that the pair $(x^*(t), c^*(t))$ is the solution to the problem (P). Therefore, optimality requires that

$$\mathcal{J}'(0) = 0\ \forall\gamma(t) \text{ and} \frac{\partial x(t,0)}{\partial \varepsilon} \forall t \in (0,T].$$

Since $\lambda(t)$ is arbitrary, $x^*(t)$ and $c^*(t)$ must satisfy

$$H_c(x^*(t), c^*(t), \lambda(t), t) = 0,$$
$$H_x(x^*(t), c^*(t), \lambda(t), t) + \dot{\lambda}(t) = 0,$$
$$\lambda(T) = \zeta.$$

The first two equations respectively correspond to (2) and (3). The third equation and the condition (12) jointly give the condition (5). Finally (4) is equivalent to the transition equation. □

From the definition of the Hamiltonian, (1),

$$(2) \Leftrightarrow F_c(x^*(t), c^*(t), t) + \lambda(t) g_c(x^*(t), c^*(t)) = 0, \tag{13}$$
$$(3) \Leftrightarrow \dot{\lambda}(t) = -\Big(F_x(x^*(t), c^*(t), t) + \lambda(t) g_x(x^*(t), c^*(t))\Big), \tag{14}$$

where

$$F_c(x(t), c(t), t) = \frac{\partial F(x(t), c(t), t)}{\partial c(t)}, \quad F_x(x(t), c(t), t) = \frac{\partial F(x(t), c(t), t)}{\partial x(t)},$$
$$g_c(x(t), c(t)) = \frac{\partial g(x(t), c(t))}{\partial c(t)}, \quad g_x(x(t), c(t)) = \frac{\partial g(x(t), c(t))}{\partial x(t)}.$$

The differential equation (3) (or equivalently (14)) is called the *Euler equaiton*, the implication of which in economic models is explained later. On the other hand, in the condition (5), $\lambda(T)x^*(T) = 0$ is called the *transversality condition* (TVC). If there is no constraint on the terminal stock $x(T)$, TVC is given by $\lambda(T) = 0$. Also its economic meaning is discussed later.

1.2 Sufficiency

When are the necessary condition of optimality both necessary and sufficient?

Theorem 2 (**Mangasarian's Sufficiency Theorem**). *Consider the problem (P). Suppose that there exists the pair* $(x^*(t), c^*(t))$ *and* $\lambda(t)$ *such that they satisfy the conditions (2)–(5). Suppose also that both of* F *and* g *satisfy Assumption 1, and they are concave with respect to* (x, c) *for all* $t \in [0, T]$*. Then* $(x^*(t), c^*(t))$ *solve the problem (P).*

Proof. [(∗) Here the argument "t" of $x(t)$ and $c(t)$ is frequently suppressed unless to do so would cause confusions.]

Define

$$D \equiv \int_0^T \Big[F(x^*, c^*, t) - F(x, c, t)\Big] dt.$$

Since $F(\cdot)$ is continuously differentiable, and concave with respect to (x, c),

$$F(x^*, c^*, t) \geq F(x, c, t) + (x^* - x)F_x(x^*, c^*, t) + (c^* - c)F_c(x^*, c^*, t) \forall t \in [0, T].$$

Substituting the above property into the definition of D yields

$$D \geq \int_0^T [(x^* - x)F_x(x^*, c^*, t) + (c^* - c)F_c(x^*, c^*, t)]dt.$$

From (13) and (14), we can arrange the above inequality as follows:

$$\begin{aligned} D &\geq \int_0^T \Big[-(x^* - x)(\dot{\lambda} + \lambda g_x(x^*, c^*, t)) - (c^* - c)\lambda g_c(s^*, c^*)\Big]\, dt \\ &= \int_0^T -\lambda\Big[(x^* - x)g_x(x^*, c^*) + (c^* - c)g_c(x^*, c^*)\Big]dt - \int_0^T (x^* - x)\dot{\lambda}dt. \end{aligned} \tag{15}$$

Integrating the last term by parts and using the TVC, we obtain

$$\begin{aligned} \int_0^T (x^* - x)\dot{\lambda}dt &= \lambda(T)(x^*(T) - x(T)) - \int_0^T \lambda(\dot{x}^* - \dot{x})dt \\ &= -\lambda(T)x(T) - \int_0^T \lambda\Big[g(x^*(t), c^*(t)) - g(x(t), c(t))\Big]dt \end{aligned} \tag{16}$$

where we used the fact that $x(0) = x^*(0)$. Substituting (16) into the right-hand-side of (15), and using the concavity of g,

$$\begin{aligned} D &\geq \int_0^T \lambda\Big[g(x^*, c^*) - g(x, c) - (x^* - x)g_x(x, c) - (c^* - c)g_c(x, c)\Big]dt + \lambda(T)x(T) \\ &\geq \lambda(T)x(T) \geq 0. \end{aligned}$$

□

Furthermore, we can show that if both of F and g are strictly concave, then $(x^*(t), c^*(t))$ is the unique solution to the problem (P).

2 Some Notes

2.1 Discounted Problem

For many problems in economics, future values of returns are discounted:

$$F(x, c, t) = \exp(-\rho t)f(x, c),$$

where $\rho > 0$ is called the *discount rate*.

Then, the problem (P) is now given by

$$\begin{aligned} \max \quad & J = \int_0^T \exp(-\rho t)f(x(t), c(t))dt \\ \text{subject to} \quad & \dot{x}(t) = g(x(t), c(t)) \quad 0 \leq t \leq T, \\ & x(T) \geq 0, \quad x(0) \text{ given.} \end{aligned} \tag{P'}$$

The Hamiltonian is given by

$$H(x(t), c(t), \lambda(t), t) = \exp(-\rho t)f(x(t), c(t)) + \lambda(t)g(x(t), c(t)).$$

Define the following new variable:

$$\mu(t) = \exp(\rho t)\lambda(t),$$

and the new function:

$$\begin{aligned}\hat{H}(x(t), c(t), \mu(t)) &= \exp(\rho t)H(x(t), c(t), \lambda(t), t)\\ &= f(x(t), c(t)) + \mu(t)g(x(t), c(t)).\end{aligned}$$

$\hat{H}$ is called the *current-value Hamiltonian*.[1]

Using this current-value Hamiltonian, we can rewrite the conditions (2)–(5) more simply:

$$\hat{H}_c(x^*, c^*, \mu) = 0 \Leftrightarrow f_c(x^*, c^*) + \mu g_c(x^*, c^*) = 0, \tag{17}$$

$$\dot{\mu} = \rho\mu - \hat{H}_x(x^*, c^*, \mu) \Leftrightarrow \dot{\mu} = \rho\mu - (f_x(x^*, c^*) + \mu g_x(x^*, c^*)) = 0, \tag{18}$$

$$\dot{x} = \hat{H}_\mu(x^*, c^*, \mu) \Leftrightarrow \dot{x} = g(x, c), \tag{19}$$

$$\mu(T)\exp(-\rho T) \geq 0,\ \ x^*(T) \geq 0,\ \ \mu(T)x^*(T)\exp(-\rho T) = 0. \tag{20}$$

2.2 Interpretations of the costate variable $\mu(t)$

As in the previous section, let $x^*(t)$ and $c^*(t)$ be the state- and control variables providing the solution to the problem (P).

Define the following function $V^* : X \to \mathbb{R}$.

$$V^*(x(0)) = \int_0^T \exp(-\rho t) f(x^*(t), c^*(t))dt. \tag{21}$$

$V^*(x(0))$ is called the *value function*, which is the maximized objective function for a given initial state $x(0)$.

Assumptiom 2. *V is continuously differentiable.*

How does the maximized value change if the initial state $x(0)$ changes?

Proposition 1. $V_x^*(x_0)\left(\equiv \dfrac{dV^*(x(0))}{dx(0)}\right) = \mu(0).$

Proof. [(∗) here the argument "t" is omitted to do so would cause confusions.]
From (21), differentiating V^* with respect to $x(0)$ yields

$$V_x^*(x(0))\left(\equiv \frac{dV^*(x(0))}{dx(0)}\right) = \int_0^T \exp(-\rho t)\left(f_x(x^*(t), c^*(t))\frac{\partial x^*(t)}{\partial x(0)} + f_c(x^*(t), c^*(t))\frac{\partial c^*(t)}{\partial x(0)}\right)dt.$$

Substituting the optimality conditions (17) and (18) into the above expression,

$$\begin{aligned}V_x^*(x(0)) &= -\int_0^T \exp(-\rho t)\left\{[\dot{\mu} + \mu\,(g_x(x^*, c^*) - \rho)]\frac{\partial x^*(t)}{\partial x(0)} + \mu g_c(x^*, c^*)\frac{\partial c^*(t)}{\partial x(0)}\right\}dt\\ &= -\int_0^T \exp(-\rho t)\left[\mu\left(g_x(x^*, c^*)\frac{\partial x^*(t)}{\partial x(0)} + g_c(x^*, c^*)\frac{\partial c^*(t)}{\partial x(0)}\right) + (\dot{\mu} - \rho\mu)\frac{\partial x^*(t)}{\partial x(0)}\right]dt\\ &= -\int_0^T \exp(-\rho t)\left[\mu\frac{\partial \dot{x}^*(t)}{\partial x(0)} + (\dot{\mu} - \rho\mu)\frac{\partial x^*(t)}{\partial x(0)}\right]dt.\end{aligned} \tag{22}$$

[1] On the other hand, H is called the *present-value Hamiltonian*.

Integrating the last term of (22) by parts yields

$$\int_0^T \exp(-\rho t)(\dot{\mu}-\rho\mu)\frac{\partial x^*(t)}{\partial x(0)}dt = \mu(T)\exp(-\rho T)\frac{\partial x^*(T)}{\partial x(0)} - \mu(0)\frac{\partial x(0)}{\partial x(0)} - \int_0^T \mu(t)\exp(-\rho t)\frac{\partial^2 x^*(t)}{\partial x(0)\partial t}dt.$$

Note that $\partial x(0)/\partial x(0) = 1$ and that the TVC (5) holds for any levels of the initial state $x(0)$. This means that the above equation is reduced to

$$\int_0^T \exp(-\rho t)[\dot{\mu}-\rho\mu]\frac{\partial x^*(t)}{\partial x(0)}dt = -\mu(0) - \int_0^T \mu(t)\exp(-\rho t)\frac{\partial \dot{x}^*(t)}{\partial x(0)}dt. \tag{23}$$

Substituting (23) into (22), we obtain

$$V_x^*(x(0)) = \mu(0).$$

□

Thus, the costate variable $\mu(0)$ is the marginal valuation in the optimal program of the state variable.

3 Dynamic Programming in Continuous Time

3.1 Bellman's Principle of Optimality

From the definition of $V^*(x(0))$ given by (21), it follows that for all $t_1 \in [0,T]$:

$$\begin{aligned} V^*(x(0)) &= \int_0^{t_1} \exp(-\rho t) f(x^*(t), c^*(t))dt + \int_{t_1}^T \exp(-\rho t) f(x^*(t), c^*(t))dt \\ &= \int_0^{t_1} \exp(-\rho t) f(x^*(t), c^*(t))dt + \exp(-\rho t_1)\int_{t_1}^T \exp[-\rho(t-t_1)] f(x^*(t), c^*(t))dt \end{aligned}$$

Now consider another problem starting from date t_1:

$$\begin{aligned} \max \quad & \int_{t_1}^T \exp[-\rho(t-t_1)] f(x(t), c(t))dt \\ \text{subject to} \quad & \dot{x}(t) = g(x(t), c(t)) \quad t_1 \le t \le T, \\ & x(T) \ge 0, \quad x(t_1) = x^*(t_1) \text{ given.} \end{aligned} \tag{P$_1$}$$

Theorem 3 (Principle of Optimality). *Suppose that the pair $(x^*(t),\, c^*(t))$ is the solution to the problem (P). Then, the following equation holds true for all $t_1 \in [0,T]$:*

$$V^*(x(0)) = \int_0^{t_1} \exp(-\rho t) f(x^*(t), c^*(t))dt + \exp(-\rho t_1) V^*(x^*(t_1)). \tag{24}$$

Proof. omitted.

Suppose we follow the solution $(x^*(t), c^*(t))$ for $0 \le t \le t_1$, and then *stop* and *reconsider* the optimal path from t_1 forward to T by solving the problem (P$_1$). The above theorem states that a solution to the problem (P$_1$) must be $(x^*(t), c^*(t))$, namely, the same as the original solution to the problem (P) on the interval from t_1 forward to T.

Note 1 (**Relationship between Value Function and Costate Variable**). *Owing to the above result, Proposition 1 holds not only for the initial date but also for any date.*

$$V_x^*(x^*(t)) = \mu(t) \forall t \in [0, T]. \tag{25}$$

Using this result, we can rewrite the TVC in (5) as follows

$$\exp(-\rho T) V_x^*(x^*(T)) x^*(T) = 0.$$

3.2 Hamilton-Jacobi-Bellman Equation

Needless to say, in (24) we can arbitrary choose the initial date, 0, and the reconsidering date t_1. Then, on the optimal path, it follows that

$$\begin{aligned} V^*(x^*(t)) &= \int_t^T \exp[-\rho(\tau - t)] f(x^*(\tau), c^*(\tau)) d\tau \\ &= \int_t^{t+\Delta t} e^{-\rho(\tau - t)} f(x^*(\tau), c^*(\tau)) d\tau + e^{-\rho \Delta t} \int_{t+\Delta t}^T e^{-\rho(\tau - t)} f(x^*(\tau), c^*(\tau)) d\tau. \end{aligned}$$

The principle of optimality implies

$$V^*(x^*(t)) = \int_t^{t+\Delta t} e^{-\rho(\tau - t)} f(x^*(\tau), c^*(\tau)) d\tau + e^{-\rho \Delta t} V^*(x^*(t + \Delta t)). \tag{26}$$

We can approximate the first integral on the right-hand-side of (26) by $f(x(t), c(t))\Delta t$. Furthermore, by Taylor-expanding the second term,

$$e^{-\rho \Delta t} V^*(x(t + \Delta t)) \simeq (1 - \rho \Delta t) V^*(x(t)) + V_x^*(x(t)) \Delta t + h.o.t,$$

where $h.o.t$ represents the collection of the higher order terms, where $(h.o.t)/\Delta t \to 0$ as $\Delta t \to 0$. Substituting this expression into (26) and taking the limit of $\Delta t \to 0$, we finally obtain

$$\rho V^*(x^*(t)) = f(x^*(t), c^*(t)) + V_x^*(x^*(t)) g(x^*(t), c^*(t)). \tag{27}$$

Briefly speaking, dynamic programming is an alternative way of solving the same problem (P) by focusing on the following functional equation with $V : X \to \mathbb{R}$ being unknown:

$$\rho V(x(t)) = \max_{c(t)} \left\{ f(x(t), c(t)) + V_x(x(t)) g(x(t), c(t)) \right\}. \tag{28}$$

(28) is called the *Hamilton-Jacobi-Bellman (HJB) equation.* To solve the problem by using the method of dynamic programming, we basically following three steps:

1. Solve the HJB equation (28) for V. The principle of optimality means that

$$V(x) = V^*(x) \forall x \in X. \tag{29}$$

   That is, the obtained function V corresponds to the value function V^*.

2. Then, from the problem given by the HJB equation, we have

$$h(x(t)) = \arg\max_{c(t)} \left\{ f(x(t), c(t)) + V_x^*(x(t)) g(x(t), c(t)) \right\},$$

where $h : X \to C$ is called the *policy function.*

3. Given $x(0)$, the pair $(x^*(t), c^*(t))$ is obtained from $\dot{x}^*(t) = g(x^*(t), h(x^*(t)))$ and $c^*(t) = h(x^*(t))$. Thus, on the optimal path, The HJB equation (28) gives (27).

Note 2 (**Relationship between Value Function and Hamiltonian).** *We consider the discounted problem. From (25), we can find that the right-hand-side of the HJB equation is the maximized current-value Hamiltonian. Namely,*

$$\rho V^*(x(t)) = \max_{c(t)} \hat{H}(x(t), c(t), \mu(t)). \tag{30}$$

4 Discounted Infinite-horizon Problem

Most economic models, including not only economic growth, but also the models of repeated games, political economy and so on, are formulated as infinite-horizon problems. Consider the following problem by taking a limit of $T \to \infty$ in the problem (P'):

$$\begin{aligned} \max \quad & J = \int_0^\infty \exp(-\rho t) f(x(t), c(t)) dt \\ \text{subject to} \quad & \dot{x}(t) = g(x(t), c(t)) \forall t \in [0, \infty), \\ & x(0) \text{ given.} \end{aligned} \tag{P$_\infty$}$$

In the same way as the proof of Theorem 2, we can show that if F and g are concave in (x, c), (2)–(4) and the TVC with $T \to \infty$ are the *sufficient conditions* of the solution, where the TVC (5) is now replaced by

$$\lim_{t \to \infty} \exp(-\rho t) \mu(t) x^*(t) = 0. \tag{31}$$

4.1 On the Necessity of the TVC in the Infinite-horizon Problem

So, the remaining issue still to be considered is on the *necessity* of the TVC in the infinite-horizon problem. Although it appears that (31) is necessary, this is not in general the case. To see why, consider the following problem which gives us a counterexample.

Example 1 (**The Problem without Discounting).**

$$\begin{aligned} \max \quad & \int_0^\infty (\ln c(t) - \ln \bar{c}) dt \\ \text{subject to} \quad & \dot{k}(t) = k(t)^\alpha - \delta k(t) - c(t) \forall t \in [0, \infty), \\ & k(0) \text{given,} \end{aligned}$$

where $c(t)$ and $k(t)$ are consumption and physical capital, respectively. $\alpha \in (0,1)$ and $\delta > 0$ is the input share of capital and the depreciation rate. $\overline{c}$ is the maximum level of consumption that can be achieved in the steady state of this model: i.e., $\overline{c} \equiv \overline{k}^{\alpha} - \delta\overline{k}$ and $\overline{k} \equiv (\alpha/\delta)^{1/(1-\alpha)}$.

Since there is no discounting in the above problem, the corresponding present value Hamiltonian always equals the current one:

$$H = \hat{H} = \ln c(t) - \ln \overline{c} + \lambda(t)(k(t)^{\alpha} - \delta k(t) - c(t)). \tag{32}$$

The first order necessary– and sufficient conditions are

$$H_c = 0 \Leftrightarrow 1/c(t) = \lambda(t), \tag{33}$$

$$\dot{\lambda} = -H_s \Leftrightarrow \dot{\lambda}(t) = -\lambda(t)(\alpha k(t)^{\alpha-1} - \delta), \tag{34}$$

$$\dot{k} = H_{\lambda} \Leftrightarrow \dot{k}(t) = k(t)^{\alpha} - \delta k(t) - c(t). \tag{35}$$

Consider the steady state where $\dot{c}(t) = 0$ and $\dot{k}(t) = 0$. Then, from (33) and (34), the steady state of $k(t)$ is given by $\overline{k} \equiv (\alpha/\delta)^{1/(1-\alpha)}$. Substituting this into (35) with $\dot{k} = 0$, the steady state of $c(t)$ is given by $\overline{c}$.

It can be verified that $(k(t), c(t)) \to (\overline{k}, \overline{c})$ as $t \to \infty$ (we shall show this fact in the next chapter). However, this condition, in turn, implies that

$$\lim_{t\to\infty} \lambda(t)k(t) = \frac{\overline{k}}{\overline{c}} > 0,$$

Thus, the optimal path does not satisfy the TVC given in the form of (31).

4.2 An Alternative TVC

The other form of the transversality condition, which always applies to the infinite-horizon problems, was shown by Michel (1982).

Theorem 4 (TVC in Infinite-Horizon Problem)**.** *Consider the problem (P_{∞}). Consider the problem (P). Suppose that this problem has an interior continuous solution $c^*(t)(0 \le t \le T)$ and the corresponding interior continuous path of the state variable $x^*(t)$. Then, there exists $\lambda(t) \ge 0$ such that $c^*(t)$ and $x^*(t)$ satisfy (2)–(4) and the transversality condition:*

$$\lim_{t\to\infty} H(x^*(t), c^*(t), \lambda(t), t) = 0,$$

or equivalently

$$\lim_{t\to\infty} \exp(-\rho t)\hat{H}(x^*(t), c^*(t), \mu(t)) = 0. \tag{36}$$

Proof. The rigorous proof is omitted. A simplified proof is found in Acemoglu (2009, Ch.7). □

4.3 The Role of Time-Discounting

Unfortunately, the TVC in the form of (36) is not easy to check. However, using a discrete-time model, Weitzman (1973) shows that the TVC in the form of (31): i.e., the limiting version of the TVC in the finite-horizon problem (5), becomes necessary when there is time-discounting and the objective function converges. Benveniste and Scheinkman (1982) shows that this result holds also in continuous-time problems. All the problems considered in this lecture assume time-discounting and converging objective functions. Thus we hereafter assume that the TVC in the form of (31) is a necessary condition for infinite-horizon problems.

References

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