1 Area minimizing surfaces

1.1 A review of surface theory.

Let $D \subset \mathbb{R}^2$ be a domain in the uv-plane and $f\colon D \to \mathbb{R}^3$ an immersion. We often refer to such an immersion as a *surface*. Then the *unit normal vector* of f is given by (with $\pm$-ambiguity)

$$(1.1)\qquad \nu := \frac{f_u \times f_v}{|f_u \times f_v|}\colon D \longrightarrow S^2 = \{\boldsymbol{x} \in \mathbb{R}^3 \,|\, |\boldsymbol{x}| = 1\} \subset \mathbb{R}^3,$$

where "$\times$" denotes the vector product of $\mathbb{R}^3$. The *first* and the *second fundamental forms* are defined as

$$(1.2)\qquad \begin{aligned} ds^2 &= df \cdot df = E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ I\!I &= -df \cdot d\nu = L\,du^2 + 2M\,du\,dv + N\,dv^2, \end{aligned}$$

where "$\cdot$" denotes the canonical inner product of $\mathbb{R}^3$. Here,

$$\begin{aligned} E &:= f_u \cdot f_u, & F &:= f_u \cdot f_v = f_v \cdot f_u, & G &:= f_v \cdot f_v, \\ L &:= -f_u \cdot \nu_u, & M &:= -f_u \cdot \nu_v = -f_v \cdot \nu_u, & N &:= -f_v \cdot \nu_v \\ &= f_{uu} \cdot \nu, & &= f_{uv} \cdot \nu, & &= f_{vv} \cdot \nu \end{aligned}$$

are called the *entries of the first and the second fundamental forms* with respect to the parameters (u, v). The *area* of the image of a compact region $\Omega \subset D$ is computed as

$$(1.3)\qquad \mathcal{A}(\Omega) := \iint_\Omega dA = \iint_\Omega |f_u \times f_v|\,du\,dv,$$

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where $dA = |f_u \times f_v|\, du\, dv = \sqrt{EG - F^2}\, du\, dv$ is said to be the *area element* of the surface.

The derivatives of ν is written as (the Weingarten Formula)

$$(1.4) \quad \nu_u = -A_1^1 f_u - A_1^2 f_v, \qquad \nu_v = -A_2^1 f_u - A_2^2 f_v,$$
$$A := \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The matrix A is called the *Weingarten matrix*, and the determinant K and the half H of the trace of A are called the *Gaussian curvature* and the *mean curvature*, respectively:

$$(1.5) \quad K := \det A = \frac{LN - M^2}{EG - F^2}, \qquad H := \frac{1}{2} \operatorname{tr} A = \frac{A_1^1 + A_2^2}{2}.$$

1.2 Area minimizing surfaces.

The purpose of this section is to show the following fact:

> For a given simple closed curve C in $\mathbb{R}^3$, the surface which minimizing area among all surfaces bounded by C is a surface whose mean curvature vanishes identically.

Setting up. As the description of the above fact is rather intuituive, we will formulate the problem.

Let C be a simple closed smooth curve in $\mathbb{R}^3$ and set

$$(1.6) \qquad \mathcal{S}_C := \left\{ f\colon \overline{D} \to \mathbb{R}^3\,;\ \begin{array}{l} f \text{ is a } C^\infty\text{-immersion} \\ f(\partial D) = C \end{array} \right\},$$

where D (resp. $\overline{D}$) is the open (resp. closed) unit disc and ∂D is its boundary:[1]

$$(1.7)\quad \overline{D} := D \cup \partial D, \qquad \begin{aligned} D &:= \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2<1\},\\ \partial D &:= \{(u,v) \in \mathbb{R}^2\,;\, u^2+v^2=1\}\\ &= \{(\cos\theta, \sin\theta)\,;\, \theta \in \mathbb{R}\}. \end{aligned}$$

Roughly speaking, $\mathcal{S}_C$ is "the set of the surfaces bounded by C". Then we set the *area functional* as

$$(1.8)\qquad \mathcal{A}\colon \mathcal{S}_C \ni f \longmapsto \mathcal{A}(f) = \iint_{\overline{D}} |f_u \times f_v|\, du\, dv.$$

Using these notations, our result can be stated as the following:

Theorem 1.1. *If a surface $f \in \mathcal{S}_C$ attains the minimum of the area functional $\mathcal{A}$, the mean curvature of f vanishes identically.*

Taking this fact into account, we define

Definition 1.2. A surface whose mean curvature vanishes identically is said to be *minimal*.

Remark 1.3. As Theorem 1.1 is a necessary condition for the minimizer, a minimal surface is not necessarily a minimizer of the area functional.

[1] A map f defined on $\overline{D}$ is said to be C^∞ if there exists a open set $\widetilde{D}$ containing $\overline{D}$ and a C^∞ map $\tilde{f}$ defined on $\widetilde{D}$ such that $\tilde{f}|_{\overline{D}} = f$.

Variations of surfaces. To show Theorem 1.1, we want to "differentiate" the functional $\mathcal{A}$.

Definition 1.4. For a surface $f \in \mathcal{S}_C$, a *variation* (fixing the boundary) of f is a C^∞-map

$$\mathcal{F}\colon \overline{D} \times (-\varepsilon, \varepsilon) \ni (u, v; t) \longmapsto f^t(u, v) := \mathcal{F}(u, v; t) \in \mathbb{R}^3$$

such that $f^0 = f$ and $f^t \in \mathcal{S}_C$ for each $t \in (-\varepsilon, \varepsilon)$, where ε is a positive number. The vector-valued function

$$V(u, v) := \left.\frac{\partial}{\partial t}\right|_{t=0} f^t(u, v) \tag{1.9}$$

is called the *variational vector field* of the variation $\mathcal{F}$.

Lemma 1.5. *For a variation $\mathcal{F} = \{f^t\}$ of $f \in \mathcal{S}_c$ with variational vector field V, it holds that*

$$\frac{d}{d\theta} f(\cos\theta, \sin\theta) \times V(\cos\theta, \sin\theta) = \mathbf{0}.$$

Proof. Since $(\cos\theta, \sin\theta)$ is a parametrization of ∂D, $\gamma^t(\theta) := f^t(\cos\theta, \sin\theta) \in C$ for all t and θ. Thus, two vectors in the left-hand side of the first assertion are both tangent to C, proving the lemma. □

The first variation formula.

Theorem 1.6. *Let $\mathcal{F} = \{f^t\}$ be a variation of $f \in \mathcal{S}_C$ with variational vector field V. Then it holds that*

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2 \iint_{\overline{D}} H\bigl(V \cdot \nu\bigr)\, dA, \tag{1.10}$$

where H, ν and dA are the mean curvature, the unit normal vector and the area element of f, respectively.

Proof. By the definition of the area (1.3), we have

$$\begin{aligned}
(*) := \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) &= \left.\frac{d}{dt}\right|_{t=0} \iint_{\overline{D}} |f_u^t \times f_v^t|\, du\, dv \\
&= \iint_{\overline{D}} \left.\frac{\partial}{\partial t}\right|_{t=0} |f_u^t \times f_v^t|\, du\, dv \\
&= \iint_{\overline{D}} \frac{\bigl(V_u \times f_v + f_u \times V_v\bigr) \cdot (f_u \times f_v)}{|f_u \times f_v|}\, du\, dv \\
&= \iint_{\overline{D}} \bigl(V_u \times f_v + f_u \times V_v\bigr) \cdot \nu\, du\, dv \\
&= \iint_{\overline{D}} \Bigl(\bigl(V_u \times f_v\bigr) \cdot \nu + \bigl(f_u \times V_v\bigr) \cdot \nu\Bigr)\, du\, dv.
\end{aligned}$$

Here, by the formula of *scalar triple product*

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b} = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}),$$

we have

$$\begin{aligned}
(*) &= \iint_{\overline{D}} \Bigl(\bigl(\nu \times f_v\bigr) \cdot V_u + \bigl(f_u \times \nu\bigr) \cdot V_v\Bigr)\; du\, dv \\
&= (\mathrm{I}) - (\mathrm{II}), \\
(\mathrm{I}) &:= \iint_{\overline{D}} \Bigl[\bigl(\bigl(\nu \times f_v\bigr) \cdot V\bigr)_u + \bigl(\bigl(f_u \times \nu\bigr) \cdot V\bigr)_v\Bigr]\; du\, dv, \\
(\mathrm{II}) &:= \iint_{\overline{D}} \Bigl[\bigl(\bigl(\nu \times f_v\bigr)_u \cdot V\bigr) + \bigl(\bigl(f_u \times \nu\bigr)_v \cdot V\bigr)\bigr)\Bigr]\; du\, dv.
\end{aligned}$$

By the Green-Stokes formula, (I) is computed as

$$
\begin{aligned}
\text{(I)} &= \iint_{\overline{D}} \left[\big((\nu \times f_v) \cdot V \big)_u - \big((\nu \times f_u) \cdot V \big)_v \right] du\, dv, \\
&= \int_{\partial D} \nu \cdot \big((f_u\, du + f_v\, dv) \times V \big) \\
&= \int_{-\pi}^{\pi} \nu \cdot \left(\frac{d}{d\theta} f(\cos\theta, \sin\theta) \times V(\cos\theta, \sin\theta) \right) d\theta = 0.
\end{aligned}
$$

Here, the last assertion is obtained by Lemma 1.5. On the other hand, using the Weingarten formula (1.4), (II) is computed as

$$
\begin{aligned}
\text{(II)} :&= \iint_{\overline{D}} \big[(\nu_u \times f_v) \cdot V + (\nu \times f_{vu}) \cdot V \\
&\qquad + (f_{uv} \times \nu) \cdot V + (f_u \times \nu_v) \cdot V \big] \, du\, dv \\
&= \iint_{\overline{D}} \big[(\nu_u \times f_v) \cdot V + (f_u \times \nu_v) \cdot V \big] \, du\, dv \\
&= -\iint_{\overline{D}} \big[\big((A_1^1 f_u + A_1^2 f_v) \times f_v \big) \cdot V \\
&\qquad + \big(f_u \times (A_2^1 f_u + A_2^2 f_v) \big) \cdot V \big] \, du\, dv \\
&= -\iint_{\overline{D}} (A_1^1 + A_2^2)(f_u \times f_v) \cdot V \, du\, dv \\
&= -\iint_{\overline{D}} 2H(\nu \cdot V) |f_u \times f_v| \, du\, dv \qquad \square
\end{aligned}
$$

Proof of Theorem 1.1. We need the following "the fundamental lemma for calculus of variations".

Lemma 1.7. *Assume a smooth function* $h\colon \overline{D} \to \mathbb{R}$ *satisifes*

$$\iint_{\overline{D}} h(u,v)\varphi(u,v)\,du\,dv = 0$$

for all C^∞*-function with* $\varphi|_{\partial D} = 0$*. Then* $h = 0$ *on* D*.*

Proof. Assume $h(u_0, v_0) > 0$ (resp. < 0)($(u_0, v_0) \in D$). By a continuity, there exists $\varepsilon > 0$ such that $h(u,v) > -$ on an ε-ball $B := B_\varepsilon(u_0, v_0)$ centered at (u_0, v_0). Let φ be a non-negative C^∞-function on $\overline{D}$ such that $\varphi > 0$ on B and 0 on $\overline{D} \setminus B$. Then

$$\iint_{\overline{D}} h\varphi\,du\,dv = \iint_B h\,\varphi\,du\,dv > 0 \qquad \text{(resp. } < 0\text{)},$$

a contradiction. □

Proof of Theorem 1.6. Assume $f \in \mathcal{S}_C$ minimizes the area. Then for any variation $\mathcal{F} = \{f^t\}$ of f, $\mathcal{A}(f^t)$ is not less than $\mathcal{A}(f) = \mathcal{A}(f^0)$. Then by Theorem 1.6, it holds that

$$0 = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}(f^t) = -2\int_{\overline{D}} H(V\cdot\nu)|f_u \times f_v|\,du\,dv.$$

Let φ be a C^∞-function on $\overline{D}$ with $\varphi|_{\partial D} = 0$. Then $f^t := f + t\varphi\nu$ is a variation of f with variational vector field $V = \varphi\nu$. Thus,

$$\iint H|f_u \times f_v|\varphi = 0.$$

Since φ is arbitrary, Lemma 1.7 yields the conclusion. □

Exercises

1-1[H] For P, Q $\in \mathbb{R}^2$, set

$$\mathcal{C}_{\mathrm{P},\mathrm{Q}} := \left\{ \gamma\colon [0,1] \to \mathbb{R}^2;\ \begin{matrix} \gamma \text{ is a regular curve} \\ \gamma(0) = \mathrm{P},\ \gamma(1) = \mathrm{Q} \end{matrix} \right\},$$

and denote by $\mathcal{L}$ the length functional:

$$\mathcal{L}(\gamma) := \int_0^1 |\dot{\gamma}(s)|\, ds \qquad \left(\dot{} = \frac{d}{ds} \right)$$

A variation of a curve $\gamma \in \mathcal{C}_{\mathrm{P},\mathrm{Q}}$ is a C^∞-map

$$\mathit{\Gamma}\colon [0,1] \times (-\varepsilon, \varepsilon) \to \gamma^t(s) = \mathit{\Gamma}(s,t) \in \mathbb{R}^2$$

such that $\gamma^t \in \mathcal{C}_{\mathrm{P},\mathrm{Q}}$ for each $t \in (-\varepsilon, \varepsilon)$ and $\gamma^0 = \gamma$.

Then show the first variation formula for the length functional

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(\gamma^t) = -\int_0^1 (V \cdot \boldsymbol{h})\, ds, \qquad \boldsymbol{h} := \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{|\dot{\gamma}|^3}(-\dot{y}, \dot{x}),$$

where V is the variational vector field of the variation $\{\gamma^t\}$ of the curve $\gamma(s) = (x(s), y(s))$.

2 Classical Examples

Graphs. For a C^∞ function $\varphi(x,y)$ on a domain (or an open set) $D \subset \mathbb{R}^2$, its graph is considered as a parametrized surface

$$(2.1)\qquad f\colon D \ni (x,y) \longmapsto \bigl(x, y, \varphi(x,y)\bigr) \in \mathbb{R}^3.$$

The surface (2.1) is minimal if and only if

$$(2.2)\quad (2\delta^3 H =)\quad (1+\varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{yy} = 0,$$

where $\delta = \sqrt{1+\varphi_x^2+\varphi_y^2}$. The (nonlinear, elliptic) partial differential equation (2.2) is called the *minimal surface equation.*

Example 2.1. A linear function $\varphi(x,y) = ax + by + c$ (a, b and c are constants) satisfies (2.2), and its graph is a plane. It is known that the *entire* (i.e., defined on whole $\mathbb{R}^2$) solution of (2.2) is a linear function (Bernstein [2-1], [2-2]).

Example 2.2. The graph of the function

$$(2.3)\quad \varphi(x,y) = \frac{1}{a}\log\frac{\cos ay}{\cos ax} \qquad (a>0 \text{ is a constant})$$

$$(x,y) \in \bigcup_{\substack{m,\, n\in\mathbb{Z}\\ m+n\colon \text{even}}} \left\{(x,y)\in\mathbb{R}^2 \;\middle|\; |ax-m\pi| < \tfrac{\pi}{2}\,,\ |ay-n\pi| < \tfrac{\pi}{2}\right\}$$

is a minimal surface, called the *Scherk surface* (Figure 1). On the domain $\{(x,y); |ax|<\pi/2, |ay|<\pi/2\}$, φ is expressed as

$$\varphi(x,y) = \tfrac{1}{a}\log\cos ax - \tfrac{1}{a}\log\cos ay.$$

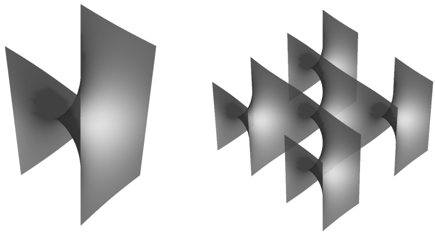

Figure 1: the Scherk surface

In general, a graph of a function $\varphi(x,y) = F(x) + G(y)$ is called a *translation surface.*

Theorem 2.3. *A translation minimal surface is congruent to a part of a plane or a part of the Scherk surface.*

Proof. For $\varphi(x,y) = F(x) + G(y)$, (2.2) is equivalent to

$$\frac{F''}{1+(F')^2} = -\frac{\ddot{G}}{1+(\dot{G})^2} =: a. \tag{2.4}$$

Since the left-hand (resp. middle) side of (2.4) is a function depending only on x (resp. y), a must be a constant. When $a = 0$, (2.4) reduce to $F'' = 0$, $\ddot{G} = 0$, i.e., φ is a linear function.

Assume $a \neq 0$. Without loss of generality, we may assume that $a > 0$ Then the first equation in (2.4) yields $\tan^{-1} F'(x) = ax + c_1$, where c_1 is a constant. By a translation along the x-axis, we can set $c_1 = 0$, and then $F(x) = -\frac{1}{a}\log\cos ax + c_2$,

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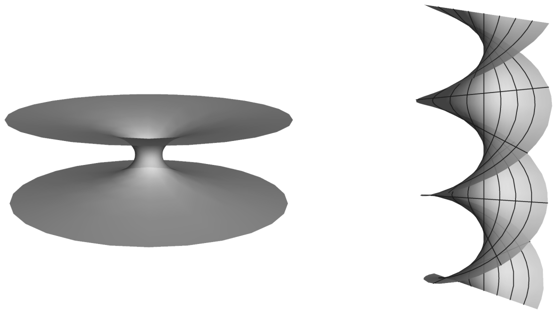

Figure 2: The catenoid and the helicoid.

with constant c_2. By a translation along the z-axis, we may set $c_2 = 0$: $F(x) = -\frac{1}{a}\log\cos ax$. Similarly, we have $G(y) = \frac{1}{a}\log\cos ay$. □

Surfaces of revolution. We consider a *surface of revolution*

$$(2.5)\quad f(u,v) = \bigl(x(u)\cos v, x(u)\sin v, z(u)\bigr),$$
$$\gamma(u) := \bigl(x(u), z(u)\bigr)\colon \mathbb{R}\supset I\to\mathbb{R}^2, \qquad x(u)\neq 0$$

where γ is a regular curve on the xz-plane, called the *profile curve* of the surface of revolution.

Example 2.4. Let $\gamma(u) = (a\cosh\frac{u}{a}, u)$, that is, γ is the graph $x = a\cosh\frac{z}{a}$ on the xz-plane, called the *catenary*. Then the surface (2.5) is minimal, called *catenoid* (Figure. 2, left).

Theorem 2.5. *A minimal surface of revolution is congruent to a part of the catenoid or the plane.*

Proof. We assume that $x(u) > 0$ and u in (2.5) is the arclength parameter of γ:

$$(x')^2 + (z')^2 = 1 \qquad (' = d/du)\,. \tag{2.6}$$

Then f is minimal if and only if

$$2H = x'z'' - z'x'' + \frac{z'}{x} = 0. \tag{2.7}$$

We shall determine $\bigl(x(u), z(u)\bigr)$ satisfying (2.7) and (2.6).

Assume $\bigl(x(u), z(u)\bigr)$ satisfy these equations and consider the case that $z' \neq 0$ for some interval I'. By a reflection about the x-axis, we may assume $z' > 0$ on I'. Differentiating (2.6), we have $x'x'' + z'z'' = 0$. Hence, noticing z' is positive on I', (2.7) is equivalent to

$$\begin{aligned} 0 &= z'\left(x'z'' - z'x'' + \frac{z'}{x}\right) = x'z'z'' - (z')^2x' + \frac{(z')^2}{x} \\ &= -x'x'x'' - \bigl(1-(x')^2\bigr)x'' + \frac{1-(x')^2}{x} = x'' + \frac{1-(x')^2}{x}. \end{aligned}$$

Since $1-(x')^2 = (z')^2 > 0$ and $x > 0$, this is equivalent to

$$\frac{-2x'x''}{1-(x')^2} = \frac{-2x'}{x}.$$

Integrating this in u, we have

$$\log\bigl(1-(x')^2\bigr) = \log x^{-2} + \text{constant}, \text{ that is, } \quad 1-(x')^2 = \frac{a^2}{x^2},$$

where a is a constant. Hence we have

$$x' = \pm\sqrt{1 - \frac{a^2}{x^2}}, \quad \text{that is,} \quad du = \frac{\pm x\,dx}{\sqrt{x^2 - a^2}}.$$

Integrating this, we get $\sqrt{x^2 - a^2} = \pm u + \text{constant}$. By a change of the arclength parameter $u \mapsto \pm u + \text{constant}$, we have

$$u = \sqrt{x^2 - a^2}, \quad \text{i.e.,} \quad x = \sqrt{u^2 + a^2}. \tag{2.8}$$

By (2.6) and the assumption $z' > 0$, we have $z' = a/\sqrt{u^2 + a^2}$, and

$$z = \int \frac{a}{\sqrt{u^2 + a^2}}\,du = a \log\left(u + \sqrt{u^2 + a^2}\right) + \text{constant}.$$

By a translation along the z-axis, we may choose the constant above to be $-a \log a$. Then we have

$$z = a \log\bigl((u + \sqrt{u^2 + a^2})/a)\bigr), \tag{2.9}$$

and thus, $\cosh \frac{z}{a} = \frac{1}{a}\sqrt{u^2 + a^2} = \frac{x}{a}$. Therefore, the curve $\bigl(x(u), z(u)\bigr)$ is a catenary, and z' does not vanish on whole I.

Otherwise, if $z' = 0$ on an interval I, $z(u)$ is constant. Thus the corresponding surface is a part of horizontal plane. □

Ruled surfaces. Let $\gamma(u)$ be a parametrized space curve, and $\xi(u)$ is a vector valued function such that $\dot\gamma(u)$, and $\xi(u)$ are linearly independent for each u. Then a parametrized surface

$$f(u, v) := \gamma(u) + v\xi(u) \tag{2.10}$$

is called a *ruled surface*, because it is a locus of moving straight lines. Replacing ξ by $\xi/|\xi|$ and $v|\xi|$ by v, we may assume without loss of generality that $|\xi| = 1$. Moreover, if we set

$$\text{(2.11)} \qquad \tilde{\gamma}(u) := \gamma(u) + \tau(u)\xi(u), \quad \tau(u) := \int_{u_0}^{u} \dot{\gamma}(t)\cdot\xi(t)\,dt,$$

(2.10) is written as $\tilde{\gamma}(u) + \tilde{v}\xi(u)$ ($\tilde{v} = v - \tau$), where $\tilde{\gamma}' \cdot \xi = 0$. Finally, we can choose u to be the arclength of γ.

Summing up, any ruled surface can be expressed as

$$\text{(2.12)} \qquad f(u,v) = \gamma(u) + v\xi(u), \qquad |\xi(u)| = |\gamma'(u)| = 1, \quad \gamma'(u)\cdot\xi(u) = 0.$$

Example 2.6. For $\gamma(u) := (0, 0, u)$ and $\xi(u) := (\cos au, \sin au, 0)$ ($a > 0$ is a constant), the surface (2.10) is minimal, called the *helicoid* (Figure 2, right).

Theorem 2.7. *A minimal ruled surface is congruent to a part of a helicoid or a plane.*

Proof. Assume that (2.12) is minimal. Since $\xi \cdot \xi' = 0$, entries of the first and the second fundamental forms satisfy $F := f_u \cdot f_v = 0$ and $N := f_{vv} \cdot \nu = 0$. Thus, f is minimal if and only if

$$2\sqrt{EG - F^2}^{\,3} H = EN - 2FM + GL = GL = 0, \text{ i.e. } \quad L = 0.$$

Since

$$|f_u \times f_v| L = (f_u \times f_v) \cdot f_{uu} \quad = \det(\gamma' + v\xi', \xi, \gamma'' + v\xi''),$$

the condition $H = 0$ is equivalent to

$$\det(\gamma', \xi, \gamma'') = 0, \tag{2.13}$$

$$\det(\xi', \xi, \gamma'') + \det(\gamma', \xi, \xi'') = 0, \tag{2.14}$$

$$\det(\xi', \xi, \xi'') = 0. \tag{2.15}$$

Here, $\{\gamma', \xi, \gamma' \times \xi\}$ forms an orthonormal basis of $\mathbb{R}^3$ for each u satisfying the following Frenet-Serret-type formulas:

$$\gamma'' = \kappa\xi, \quad \xi' = -\kappa\gamma' + \tau(\gamma' \times \xi), \quad (\gamma' \times \xi)' = -\tau\xi, \tag{2.16}$$

where κ and τ are smooth functions in u. In fact, since $|\gamma'| = 1$, $\gamma'' \cdot \gamma' = 0$, and (2.13) implies $\gamma'' \cdot (\gamma' \times \xi) = 0$. Thus the first equation follows. Similarly, $\xi' \cdot \xi = 0$ and $\xi' \cdot \gamma' = (\xi \cdot \gamma')' - \xi \cdot \gamma'' = -\xi \cdot \gamma'' = -\kappa$ yield the second equation. Finally,

$$(\gamma' \times \xi)' \cdot \gamma' = -(\gamma' \times \xi) \cdot \gamma'' = 0, \quad (\gamma' \times \xi)' \cdot \xi = -(\gamma' \times \xi) \cdot \xi' = -\tau$$

imply the third equation.

Differentiating (2.14) with (2.16), we have

$$\xi'' = -\kappa'\gamma' - (\kappa^2 + \tau^2)\xi + \tau'(\gamma' \times \xi). \tag{2.17}$$

Hence (2.14), $0 = \det(\gamma', \xi, \xi'') = \tau'$, and then τ is constant. In addition, by (2.15), we have

$$0 = \det(\xi', \xi, \xi'') = (-\kappa\tau' + \kappa'\tau) = \det(\gamma', \xi, \gamma' \times \xi) = \kappa'\tau.$$

Assume the constant $\tau \neq 0$. Then $\kappa' = 0$, that is, κ is also constant, and (2.17) turns to be

$$\xi'' = -(\kappa^2 + \tau^2)\xi. \tag{2.18}$$

So, if we set $\tilde{\gamma} := \gamma + (\kappa^2 + \tau^2)\xi$ and $\tilde{v} = v - (\kappa^2 + \tau^2)$, we have $f = \tilde{\gamma} + \tilde{v}\xi$ with $\tilde{\gamma}'' = 0$, that is, $\tilde{\gamma}$ is a straight line. Then by an isometry of $\mathbb{R}^3$ and a change of parameter u, we can set $\tilde{\gamma}(u) = (0, 0, u)$. Since ξ is perpendicular to $\tilde{\gamma}' = (0, 0, 1)$, the image of $\xi(u)$ lies on the unit circle in the xy-plane. Hence, by (2.18), up to an isometry and a change of parameters, we have

$$\xi(u) = (\cos au, \sin au, 0), \qquad a = \sqrt{\kappa^2 + \tau^2} > 0,$$

Then the surface is a helicoid.

On the other hand, when $\tau = 0$, $\gamma' \times \xi$ is constant, and we may set $\gamma' \times \xi = (0, 0, 1)$. Since γ' and ξ are perpendicular to $(0, 0, 1)$, $f(u, v) = \gamma(u) + v\xi(u)$ lies on a plane parallel to the xy-plane, that is, the image of the surface is part of a plane. □

References

[2-1] Bernstein, S. N., *Sur une théorme de géometrie et ses applications aux équations dérivées partielles du type elliptique*, Comm. Soc. Math. Kharkov **15** 38–45. (1915–1917).

[2-2] Osserman, R., A SURVEY OF MINIMAL SURFACES, Dover Publ.

Exercises

2-1[H] Show that the surface $\{(x, y, z)\,;\, \sinh x \sinh y = \sin z\}$ is minimal.

3 Isothermal Coordinates

A Review of Complex Analysis. Let $\mathbb{C}$ be the complex plane. A C^1-function[2] $f\colon \mathbb{C} \ni D \in z \mapsto w = f(z) \in \mathbb{C}$ defined on a domain D is said to be *holomorphic* if the derivative

$$f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists for all $z \in D$.

Fact 3.1 (The Cauchy-Riemann equation). *A function $f\colon \mathbb{C} \ni D \to \mathbb{C}$ is holomorphic if and only if*

$$\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta} \qquad \textit{and} \qquad \frac{\partial u}{\partial \eta} = -\frac{\partial v}{\partial \xi} \tag{3.1}$$

holds on D, where $w = f(z)$, $z = \xi + i\eta$, $w = u + iv$ ($i = \sqrt{-1}$).

For functions of complex variable $z = \xi + i\eta$, we set

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right). \tag{3.2}$$

Corollary 3.2. *For a complex function f, (3.1) is equivalent to*

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{3.3}$$

Proof. Setting $w = f(z) = u + iv$ and $z = \xi + i\eta$. Then the real (resp. imaginary) part of the left-hand side of (3.3) coincides with the first (resp. second) equation of (3.1). □

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[2]Of class C^1 as a map from $D \subset \mathbb{R}^2$ to $\mathbb{R}^2$.

Definition 3.3. A real-valued function $\varphi\colon \mathbb{R}^2 \supset U \to \mathbb{R}$ is said to be *harmonic* if it satisfies the Laplace equation

$$\Delta\varphi := \varphi_{\xi\xi} + \varphi_{\eta\eta} = 0.$$

Lemma 3.4. *If a function $\varphi\colon \mathbb{C} \supset D \to \mathbb{R}$ is harmonic, $\partial\varphi/\partial z$ is a holomorphic function on D, where z is a complex coordinate of $\mathbb{C}$.*

Proof. Corollary 3.2 yields the conclusion since

$$\frac{\partial}{\partial \bar{z}}\frac{\partial\varphi}{\partial z} = \frac{\partial^2\varphi}{\partial\bar{z}\partial z} = \frac{1}{4}\Delta\varphi. \qquad \square$$

Isothermal Coordinates.

Definition 3.5. Let $f\colon M^2 \to \mathbb{R}^3$ be an immersion of 2-manifold, and ds^2 its first fundamental form. A local coordinate chart $\bigl(U;(u,v)\bigr)$ of M^2 is called an *isothermal coordinate system* or a *conformal coordinate system* if ds^2 is written in the form[3]

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \sigma = \sigma(u,v) \in C^\infty(U).$$

Example 3.6. A parametrization of the catenoid in Example 2.4 is isothermal if $a = 1$. In fact, the first fundamental form is expressed as $\cosh^2(u/a)(du^2 + a^2 dv^2)$.

[3]The notion of the isothermal coordinate system can be defined not only for surfaces but also for Riemannian 2-manifolds, that is, differentiable 2-manifolds M^2 with Riemannian metrics ds^2 (the first fundamental forms).

Definition 3.7. Two charts $\big(U_j;(u_j,v_j)\big)$ $(j=1,2)$ of a 2-manifold M^2 has the *same* (resp. *opposite*) *orientation* if the Jacobian $\frac{\partial(u_2,v_2)}{\partial(u_1,v_1)}$ is positive (resp. negative) on $U_1 \cap U_2$. A manifold M^2 is said to be *oriented* if there exists an atlas $\big\{\big(U_j;(u_j,v_j)\big)\big\}$ such that all charts have the same orientations. A choice of such an atlas is called an *orientation* of M^2.

Proposition 3.8. *Let (u,v) be an isothermal coordinate system of a surface. Then another coordinate system (ξ,η) is also isothermal if and only if the parameter change $(\xi,\eta)\mapsto(u,v)$ satisfy*

$$\frac{\partial u}{\partial \xi}=\varepsilon\frac{\partial v}{\partial \eta},\qquad \frac{\partial u}{\partial \eta}=-\varepsilon\frac{\partial v}{\partial \xi}, \tag{3.4}$$

where $\varepsilon=1$ (resp. -1) if (u,v) and (ξ,η) has the same (resp. the opposite) orientation.

Proof. If we write $ds^2=e^{2\sigma}(du^2+dv^2)$, it holds that

$$ds^2=e^{2\sigma}\big((u_\xi^2+v_\xi^2)\,d\xi^2+2(u_\xi v_\eta+u_\eta v_\xi)\,d\xi\,d\eta+(u_\eta^2+v_\eta^2)\,d\eta^2\big).$$

Thus, (ξ,η) is isothermal if and only if

$$u_\xi^2+v_\xi^2=u_\eta^2+v_\eta^2,\qquad (u_\xi v_\eta+u_\eta v_\xi)=0. \tag{3.5}$$

The second equality yields $(v_\xi,v_\eta)=\varepsilon(-u_\eta,u_\xi)$ for some function ε. Substituting this into the first equation of (3.5), we get $\varepsilon=\pm1$. Moreover,

$$\frac{\partial(u,v)}{\partial(\xi,\eta)}=\det\begin{pmatrix}u_\xi & u_\eta\\ v_\xi & v_\eta\end{pmatrix}=\det\begin{pmatrix}u_\xi & u_\eta\\ -\varepsilon u_\eta & \varepsilon u_\xi\end{pmatrix}=\varepsilon(u_\xi^2+u_\eta^2).$$

Thus, the conclusion follows. □

Corollary 3.9. *Let (u, v) is an isothermal coordinate system. Then a coordinate system (ξ, η) is isothermal and has the same orientation as (u, v) if and only if the map $\xi + i\eta \mapsto u + iv$ $(i = \sqrt{-1})$ is holomorphic.*

Proof. Equations 3.4 for $\varepsilon = +1$ are nothing but the Cauchy-Riemann equations (3.1). □

Fact 3.10 (Section 15 in [3-1])**.** *Let (M^2, ds^2) be an arbitrary Riemannian manifold. Then for each $p \in M^2$, there exists an isothermal chart containing p.*

Corollary 3.11. *Any oriented Riemannian 2-manifold (M^2, ds^2) has a structure of Riemann surface (i.e., a complex 1-manifold) such that for each complex coordinate $z = u + iv$, (u, v) is an isothermal coordinate system for ds^2.*

Proof. Let $p \in M^2$ and take a local coordinate chart $\bigl(U_p; (x, y)\bigr)$ at p which is compatible to the orientation of M^2. Then by Fact 3.10, their exists a isothermal coordinate system $\bigl(V_p; (u_p, v_p)\bigr)$ at p. Moreover, replacing (u, v) by (v, u) if necessary, we can take (u, v) which has the same orientation of (x, y). Thus, we have an atlas $\bigl\{\bigl(V_p; (u_p, v_p)\bigr)\bigr\}$ consists of isothermal coordinate systems. Since each chart is compatible of the orientation, the coordinate change $z_p = u_p + iv_p \mapsto u_q + iv_q = z_q$ is holomorphic. Hence we get a complex atlas $\bigl\{\bigl(V_p; z_p\bigr)\bigr\}$. □

Isothermal Coordinates for Minimal surfaces. Though existence of isothermal parameters are guaranteed as Fact 3.10, we shall give an alternative proof of it for minimal surfaces. The proof is due to [3-2].

Lemma 3.12 (The Poincaré lemma [Theorem 12.2 in [3-1]])**.** *Let $D \subset \mathbb{R}^2$ be a simply connected domain, and let λ, μ be smooth functions defined on D. If*

$$\lambda_\xi = \mu_\eta, \qquad \text{that is} \qquad d\omega = 0 \quad \text{for} \quad \omega = \lambda\, d\xi + \mu\, d\eta,$$

then there exists a smooth function α on D such that

$$\alpha_\xi = \lambda, \quad \alpha_\eta = \mu, \qquad \text{that is,} \qquad d\alpha = \omega.$$

Proposition 3.13. *Assume that the graph of $\varphi\colon D_R \to \mathbb{R}$ defined on a disc $D_R := \{(x,y)\,;\, x^2+y^2 < R^2\}$ is minimal surface. Then there exists smooth map*

$$X\colon D_R \ni (x,y) \longmapsto \big(\xi(x,y), \eta(x,y)\big) \in X(D_R) \subset \mathbb{R}^2$$

such that

1. *$X\colon D_R \to X(D_R)$ is a diffeomorphism with $X(\mathbf{0}) = \mathbf{0}$,*
2. *(ξ,η) is an isothermal parameter of the graph $z = \varphi(x,y)$.*
3. *$X(D_R) \supset \{(\xi,\eta)\,;\, \xi^2+\eta^2 < R^2\}$.*

Proof. By the assumption, φ satisfies (2.2):

$$(1+\varphi_x^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_y^2)\varphi_{yy} = 0. \tag{3.6}$$

Let $W := \sqrt{1+\varphi_x^2+\varphi_y^2}$ and set

$$\lambda_1 := \frac{1+\varphi_x^2}{W}, \quad \mu_1 = \lambda_2 := \frac{\varphi_x\varphi_y}{W}, \quad \mu_2 := \frac{1+\varphi_y^2}{W}. \tag{3.7}$$

So one can show that $(\lambda_1)_y = (\mu_1)_x$ and $(\lambda_2)_y = (\mu_2)_x$. Then by Lemma 3.12, there exist smooth functions α, β such that

$$\alpha_x = \lambda_1, \quad \alpha_y = \mu_1, \quad \beta_x = \lambda_2, \quad \beta_y = \mu_2.$$

Adding constants, we may assume $\alpha(0,0) = \beta(0,0) = 0$. Using these, we define a map $X = (\xi, \eta)\colon D_R \to \mathbb{R}^2$ by

$$\xi(x,y) := x + \alpha(x,y), \qquad \eta(x,y) := y + \beta(x,y). \tag{3.8}$$

By definition, the Jacobian of X is computed as

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \det \begin{pmatrix} 1+\lambda_1 & \mu_1 \\ \lambda_2 & 1+\mu_2 \end{pmatrix} = 2(2 + \varphi_x^2 + \varphi_y^2) > 0.$$

Hence X is a local diffeomorphism. So, to prove (1), it is sufficient to show that X is injective: Fix $\boldsymbol{x}_0 = (x_0, y_0) \in D_R$ and $\boldsymbol{h} = (h,k)$ such that $\boldsymbol{x}_1 := \boldsymbol{x}_0 + \boldsymbol{h} \in D_R$. We set $\boldsymbol{x}_t := \boldsymbol{x} + t\boldsymbol{h}$ $(0 \leqq t \leqq 1)$, $\boldsymbol{X}_t := X(\boldsymbol{x}_t)$, $\boldsymbol{\alpha}_t := \bigl(\alpha(\boldsymbol{x}_t), \beta(\boldsymbol{x}_t)\bigr)$, and

$$q(t) := \boldsymbol{h} \cdot (\boldsymbol{\alpha}_t - \boldsymbol{\alpha}_0) \quad (0 \leqq t \leqq 1).$$

Then by the mean value theorem, it holds that

$$\begin{aligned} \boldsymbol{h} \cdot (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) &= q'(\tau) = \boldsymbol{h} \cdot \boldsymbol{\alpha}'(\tau) = h^2\lambda_1 + hk(\mu_1 + \lambda_2) + k^2\mu_2 \\ &= W^{-1}\bigl((1+\varphi_x^2)h^2 + 2\varphi_x\varphi_y hk + (1+\varphi_y^2)k^2\bigr) > 0 \end{aligned}$$

for some $\tau \in (0,1)$, because the quadratic form in (h,k) of the right-hand side is positive definite. Hence

$$\begin{aligned} |X(\boldsymbol{x}_0 + \boldsymbol{h}) - X(\boldsymbol{x}_0)|^2 &= |\boldsymbol{x}_1 - \boldsymbol{x}_0 + \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0|^2 \\ &= |\boldsymbol{h}|^2 + 2\boldsymbol{h} \cdot (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) + |\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0|^2 \geqq |\boldsymbol{h}|^2, \end{aligned} \tag{3.9}$$

which proves the injectivity of X.

By definition, $d\xi = (1+\lambda_1)\,dx + \mu_1\,dy$, and $d\eta = \lambda_2\,dx + (1+\mu_2)\,dy$ hold. So,

$$(3.10)\quad d\xi^2 + d\eta^2 = \left(1+\frac{1}{W}\right)^2 ds^2,$$
$$ds^2 = (1+\varphi_x^2)\,dx^2 + 2\varphi_x\varphi_y\,dx\,dy + (1+\varphi_y^2)\,dy^2,$$

proving (2).

Finally, we prove (3). Let $\rho := \inf\{|\boldsymbol{X}|\,|\,\boldsymbol{X} \in X(D_R)^c\}$. Then $\rho > 0$ because X is a diffeomorphism and $X(\mathbf{0}) = \mathbf{0}$. Since the result is obvious if $\rho = +\infty$, we consider the case $\rho \in (0,\infty)$. The set $X(D_R)^c$ is a closed subset in $\mathbb{R}^2$ because X is a diffeomorphism. Hence there exists $\boldsymbol{X}_\rho \in X(D_R)^c$ with $|\boldsymbol{X}_\rho| = \rho$. Since $\boldsymbol{X}_\rho \in \partial X(D_R)^c = \partial X(D_R)$, there exists a sequence $\{\boldsymbol{X}_n\} \subset X(D_R)$ which convergences to $\boldsymbol{X}_\rho$. The inverse image of $\{\boldsymbol{x}_n := X^{-1}(\boldsymbol{X}_n)\}$ of such a sequence is a sequence in D_R, which does not accumlate in D_R. Hence, by taking a subsequence if necessary, $\{\boldsymbol{x}_n\}$ converges to $\boldsymbol{x}_R \in \partial D_R$, that is, $|\boldsymbol{x}_R| = R$. Here, setting $\boldsymbol{x}_0 = (0,0)$ in (3.9), we have $|\boldsymbol{x}_n| \leqq |\boldsymbol{X}_n|$, and then, $|\boldsymbol{X}_\rho| \geqq R$, that is, $X(D_R)^c \subset D_R^c$, proving (3). □

The minimal surface equation. The equation for minimal surfaces are linearlized by the isothermal coordinate system:

Proposition 3.14. *Let $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ be a surface, and assume the parameter (u,v) is isothermal. Then f is minimal if and only if $\Delta f = f_{uu} + f_{vv} = 0$.*

Proof. Write the first fundamental form as $ds^2 = e^{2\sigma}(du^2+dv^2)$. Then $f_u \cdot f_u = f_v \cdot f_v = e^{2\sigma}$ and $f_u \cdot f_v = 0$ hold. So

$$f_{uu} \cdot f_u = \frac{1}{2}(f_u \cdot f_u)_u = \sigma_u e^{2\sigma},$$
$$f_{vv} \cdot f_u = (f_v \cdot f_u)_v - f_v \cdot f_{vu} = -\frac{1}{2}(f_v \cdot f_v)_u = -\sigma_u e^{2\sigma},$$

that is $(f_{uu} + f_{vv}) \cdot f_u = 0$. Similarly, one can show $(f_{uu} + f_{vv}) \cdot f_v = 0$ and hence $f_{uu} + f_{vv}$ is parallel to the unit normal vector ν. On the other hand, the mean curvature H is computed as

$$H = \frac{L+N}{2e^{2\sigma}} = \frac{(f_{uu}+f_{vv})\cdot \nu}{2e^{2\sigma}}, \quad \text{that is,} \quad \varDelta f = 2He^{2\sigma}\nu. \quad \square$$

References

[3-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版）.

[3-2] Osserman, R., A SURVEY OF MINIMAL SURFACES, Dover Publ.

Exercises

3-1^H Consider two minimal surfaces

$$f(u,v) = (\cosh u \cos v, \cosh u \sin v, u),$$
$$g(s,t) = (s\cos t, s\sin t, t).$$

(1) Show that (u,v) is an isothermal parameter of f.

(2) Show that there exists a isothermal parameter (u,v) of g.

4 Bernstein's Theorem

More complex analysis.

Theorem 4.1 (Liouville's theroem). *A bounded holomorphic function defined on the whole complex plane $\mathbb{C}$ is constant.*

Proof. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| \leqq M$ for every $z \in \mathbb{C}$. Fix a point $z \in \mathbb{C}$. Then by Cauchy's integral formula, it holds that

$$f'(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\,d\zeta}{(z-\zeta)^2} \quad (C_R : \zeta = z + Re^{i\theta}; -\pi < \theta \leqq \pi),$$

where R is an arbitrary positive number. Hence

$$\begin{aligned} |f'(z)| &\leqq \frac{1}{2\pi}\int_{C_R} \frac{|f(\zeta)|\,|d\zeta|}{|z-\zeta|^2} \\ &\leqq \frac{1}{2\pi}\int_{C_R} \frac{M\,|d\zeta|}{|z-\zeta|^2} = \frac{1}{2\pi}\int_{\pi}^{\pi} \frac{M\,R\,d\theta}{R^2} = \frac{M}{R}. \end{aligned}$$

Since R is arbitrary, we can conclude $f'(z) = 0$ by letting $R \to \infty$. Moreover, since z is arbitrary, $f'(z) = 0$ holds on $\mathbb{C}$, proving that f is constant. □

Corollary 4.2. *A holomorphic function defined on $\mathbb{C}$ into the upper-half plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ must be constant.*

15. July, 2016.

Proof. Note that a linear fractional transformation

$$F(z) = \frac{z-i}{z+i} \qquad (i = \sqrt{-1})$$

maps the upper-half plane H to the unit disc $D = \{w \in \mathbb{C} \mid |w| < 1\}$ bijectively. Then for each holomorphic function $f\colon \mathbb{C} \to H$, $F \circ f$ is a bounded holomorphic function defend on $\mathbb{C}$. □

Conformal minimal surfaces. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion, where Σ is an orientable 2-dimensional manifold. As seen in Corollary 3.11, there exists a structure of Riemann surface such that each complex coordinate $z = u + iv$ gives an isothermal coordinate system.

Definition 4.3. An immersion $f\colon \Sigma \to \mathbb{R}^3$ of a Riemann surface Σ is said to be *conformal* if each complex coordinate $z = u+iv$ is isothermal.

In this section, we consider conformal minimal immersions $f\colon \Sigma \to \mathbb{R}^3$. Then by virtue of Proposition , and Lemma 3.4,

$$\phi := \frac{\partial f}{\partial z} \left(= \frac{1}{2}\left(\frac{\partial f}{\partial u} - i\frac{\partial f}{\partial v}\right)\right) : \Sigma \to \mathbb{C}^3 \tag{4.1}$$

is holomorphic for each complex coordinate $z = u + iv$ of Σ. Moreover, we have

Proposition 4.4. *Let $f\colon \Sigma \to \mathbb{R}^3$ be a conformal minimal immersion. Then for each complex coordinate chart $(U; z = u+iv)$*

of Σ, ϕ *in* (4.1) *satisfies*

$$(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = 0, \tag{4.2}$$

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0, \tag{4.3}$$

where we write $\phi = (\phi_1, \phi_2, \phi_3)$.

Proof. Since $\phi = (1/2)(f_u - i f_v)$,

$$\begin{aligned}(\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = \phi \cdot \phi &= \frac{1}{4}\bigl(f_u \cdot f_u - f_v \cdot f_v - 2i f_u \cdot f_v\bigr) \\ &= \frac{1}{4}\bigl((E - G) - 2iF\bigr) = 0,\end{aligned}$$

where E, F and G are the components of the first fundamental form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = E(du^2 + dv^2)$. Then (4.2) follows. On the other hand,

$$\begin{aligned}|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = \phi \cdot \bar{\phi} &= \frac{1}{4}\bigl(f_u \cdot f_u + f_v \cdot f_v\bigr) \\ &= \frac{1}{4}(E + G) = \frac{E}{2} > 0,\end{aligned}$$

proving (4.3). □

Bernstein's Theorem We prove the following global result of minimal surfaces:

Theorem 4.5 (Bernstein, 1915)**.** *Let* $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ *be a smooth function defined on the whole plane* $\mathbb{R}^2$*, and assume the graph of* φ *is minimal surface. Then* $\varphi(x, y)$ *is a linear function in* (x, y)*. In other words, the only entire minimal graphs are planes.*

Proof. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a solution of the minimal surface equation

$$(1+\varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1+\varphi_x^2)\varphi_{yy} = 0. \tag{4.4}$$

Then there exists functions ξ and η satisfying

$$d\xi = \left(1 + \frac{1+\varphi_x^2}{W}\right) dx + \frac{\varphi_x\varphi_y}{W}\, dy, \tag{4.5}$$

$$d\eta = \frac{\varphi_x\varphi_y}{W}\, dx + \left(1 + \frac{1+\varphi_y^2}{W}\right) dy, \tag{4.6}$$

where $W = \sqrt{1+\varphi_x^2+\varphi_y^2}$. Moreover, by Proposition 3.13, we know that the map

$$\mathbb{R}^2 \ni (x,y) \longmapsto (\xi,\eta) \in \mathbb{R}^2$$

is a diffeomorphism and

$$f\colon \mathbb{C} \ni \zeta := \xi + i\eta \longmapsto \bigl(x(\xi,\eta), y(\xi,\eta), \varphi(x(\xi,\eta), y(\xi,\eta))\bigr) \in \mathbb{R}^3,$$

is a conformal reparametrization of the graph of φ. We let ϕ as in (4.1):

$$\phi = (\phi_1,\phi_2,\phi_3) = \frac{\partial f}{\partial \zeta} = \left(\frac{\partial x}{\partial \zeta}, \frac{\partial y}{\partial \zeta}, \frac{\partial \varphi}{\partial \zeta}\right), \qquad (\zeta = \xi + i\eta).$$

Since

$$
(4.7)\qquad 4\,\mathrm{Im}\left(\phi_1\overline{\phi}_2\right) = 4\,\mathrm{Im}\bigl(x_\zeta\overline{y_\zeta}\bigr) = \mathrm{Im}(x_\xi - ix_\eta)(y_\xi + iy_\eta)
$$

$$
= x_\xi y_\eta - y_\xi x_\eta = \det\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}^{-1}
$$

$$
= \left(1 + \frac{1+\varphi_x^2}{W}\right)\left(1 + \frac{1+\varphi_y^2}{W}\right) - \frac{\varphi_x^2\varphi_y^2}{W^2} > 0,
$$

both ϕ_1 and ϕ_2 never vanish, and

$$
\mathrm{Im}\,\frac{\phi_1}{\phi_2} = \frac{\mathrm{Im}\,\phi_1\overline{\phi_2}}{|\phi_2|^2} > 0.
$$

Then we have a holomorphic map of $\mathbb{C}$ into the upper half plane

$$
\frac{\phi_1}{\phi_2} : \mathbb{C} \longrightarrow H.
$$

Hence by Liouville's Theorem 4.1, we conclude that

$$
(4.8)\quad \phi_1 = a\phi_2, \qquad \text{that is} \qquad \frac{\partial x}{\partial\zeta} = a\frac{\partial y}{\partial\zeta} \qquad (a \in \mathbb{C}\setminus\{0\}).
$$

Moreover, by (4.7), we have

$$
(4.9)\qquad \mathrm{Im}(\phi_1\overline{\phi_2}) = \mathrm{Im}(a|\phi_2|^2) > 0, \quad \text{that is,} \quad \mathrm{Im}\,a > 0.
$$

By (4.8), and noticing x and y are real valued functions, we have

$$
\frac{\partial x}{\partial\bar\zeta} = \overline{\frac{\partial x}{\partial\zeta}} = \overline{a\frac{\partial y}{\partial\zeta}} = \bar a\frac{\partial y}{\partial\bar\zeta}.
$$

Then, if we set $w = x + iy$,

$$\frac{\partial w}{\partial \bar{\zeta}} = \frac{\partial x}{\partial \bar{\zeta}} + i\frac{\partial y}{\partial \bar{\zeta}} = (\bar{a}+i)\frac{\partial y}{\partial \bar{\zeta}}, \quad \frac{\partial \bar{w}}{\partial \bar{\zeta}} = \frac{\partial x}{\partial \bar{\zeta}} - i\frac{\partial y}{\partial \bar{\zeta}} = (\bar{a}-i)\frac{\partial y}{\partial \bar{\zeta}}$$

hold. We set

$$q := q(\zeta) = (-\bar{a}+i)w + (\bar{a}+i)\bar{w}, \quad \bigl(w(\zeta) = x(\zeta)+iy(\zeta)\bigr). \tag{4.10}$$

Then we have

$$\frac{\partial q}{\partial \bar{\zeta}} = (-\bar{a}+i)(\bar{a}+i)\frac{\partial y}{\partial \bar{\zeta}} + (\bar{a}+i)(\bar{a}-i)\frac{\partial y}{\partial \bar{\zeta}} = 0,$$

that is, $\zeta \mapsto q$ is a holomorphic function. If we write $q = u + iv$ and $a = s + it$, we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -2t \\ 2 & -2s \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (t = \operatorname{Im} a > 0). \tag{4.11}$$

that is, x and y are linear functions of u and v.

By holomorphicity of w, (u, v) is also an isothermal parameter of the surface. We set

$$\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2, \tilde{\phi}_3) := \left(\frac{\partial x}{\partial w}, \frac{\partial y}{\partial w}, \frac{\partial z}{\partial w}\right).$$

Since x and y are linear functions of u and v, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are constants. On the other hand, since w is an isothermal (complex) parameter, (4.2) holds for $\tilde{\phi}$:

$$\tilde{\phi}_3^2 = -\tilde{\phi}_1^2 - \tilde{\phi}_2^2 = \text{constant}.$$

Therefore, the third coordinate z is also a liner function of u and v. Hence

$$z(u,v) = \varphi\bigl(x(u,v), y(u,v)\bigr)$$

is a liner function in (u,v). Thus, by (4.11), $\varphi(x,y)$ is a linear function. □

References

[4-1] Osserman, R., A survey of minimal surfaces, Dover Publ.

Exercises

Solve one of the following problems:

4-1H Let $f\colon \mathbb{C} \subset U \to \mathbb{R}^3$ be a conformal minimal immersion and set $\phi = (\phi_1, \phi_2, \phi_3)$ as (4.1). Show that

(1) the first fundamental form of f is expressed as

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \qquad \text{where } e^{2\sigma} = 2(|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2),$$

(2) the unit normal vector field ν is expressed as

$$\begin{aligned}\nu &= \frac{f_u \times f_v}{|f_u \times f_v|} \\ &= \frac{-i(\phi_2\overline{\phi_3} - \phi_3\overline{\phi_2}, \phi_3\overline{\phi_1} - \phi_1\overline{\phi_3}, \phi_1\overline{\phi_2} - \phi_2\overline{\phi_1})}{|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2},\end{aligned}$$

(3) and the composition of $\nu\colon U \to S^2$ with the stereographic projection

$$\pi \circ S^2 \ni (\nu_1, \nu_2, \nu_3) \longmapsto \frac{1-\nu_3}{\nu_1 + i\nu_2} \in \mathbb{C} \cup \{\infty\}$$

is expressed as

$$\pi \circ \nu = \frac{\phi_3}{\phi_1 - i\phi_2},$$

here $z = u + iv$ is the complex coordinate of U. (Hint: $\phi_3^2 = -(\phi_1 + i\phi_2)(\phi_1 - i\phi_2)$.)

4-2[H] Find a non-trivial (non-linear) solution $\varphi(x, y)$ of the partial differential equation

$$(1-\varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1-\varphi_x^2)\varphi_{yy} = 0,$$

which is defined on whole $\mathbb{R}^2$ (Hint: Try a similar method as in 2).