## 7 Surfaces of constant negative curvatureBäcklund Transformations

Bäcklund transformations. Let $f: D \rightarrow \mathbb{R}^{3}$ be a surface with unit normal vector field $\nu: D \rightarrow \mathbb{R}^{3}$. A surface $\hat{f}: D \rightarrow \mathbb{R}^{3}$ is said to be a Bäcklund transformation of $f$ if there exists a positive constant $r$ and an angle $\delta$ such that
(B-1) $\hat{f}(p)-f(p)$ is a tangent vector of both the surface $f$ and $\hat{f}$ at $p \in D$,
(B-2) $|f(p)-\hat{f}(p)|=r$,
(B-3) the angle between $\nu(p)$ and $\hat{\nu}(p)$ is $\delta$,
for each $p \in D$.
The following proposition gives a necesary condition for existence of Bäcklund transformations:
Proposition 7.1. Assume hat there exists a Bäcklund transformation $\hat{f}$ of an immersion $f: D \rightarrow \mathbb{R}^{3}$. Then $r$ and $\delta$ in ( $B-1$ ) and ( $B-3$ ) satisfy

$$
K=-\frac{\sin ^{2} \delta}{r^{2}}
$$

where $K$ is the Gaussian curvature, that is, $K$ is negative constant. Moreover, the Gaussian curvature $\hat{K}$ of the Bäcklund transformation $\hat{f}$ is the same constant as $f$.

To have constant negative Gaussian curvature is also the sufficient condition for existence of Bäcklund transformations:

[^0]Proposition 7.2. Let $f: D \rightarrow \mathbb{R}^{3}$ be a smooth map with unit normal vector field $\nu$, where $D$ is a simply connected domain of the uv-plane, and

$$
\begin{aligned}
d s^{2} & :=d f \cdot d f=d u^{2}+2 \cos \theta d u d v+d v^{2} \\
I I & :=-d f \cdot d \nu=2 \sin \theta d u d v
\end{aligned}
$$

where $\theta=\theta(u, v)$ is a smooth function satisfying the sine-Grodon equation

$$
\theta_{u v}=\sin \theta .
$$

We fix $p_{0}=\left(u_{0}, v_{0}\right) \in D$ and $\delta \in(0, \pi)$. Then
(B-1) for any $\varphi_{0} \in \mathbb{R}$, there exists a unique solution of the differential equation

$$
(\varphi-\theta)_{u}=2 \cot \frac{\delta}{2} \sin \frac{\varphi+\theta}{2}, \quad(\varphi+\theta)_{v}=2 \tan \frac{\delta}{2} \sin \frac{\varphi-\theta}{2}
$$

with initial condition $\varphi\left(u_{0}, v_{0}\right)=\varphi_{0}$,
(B-2) for $\varphi$ in (B-1), let

$$
\begin{aligned}
& \hat{f}:=f+\sin \delta\left(\cos \frac{\varphi}{2} e_{1}+\sin \frac{\varphi}{2} e_{2}\right) \\
& \hat{\nu}:=\cos \delta \nu-\sin \delta-\sin \delta\left(\left(-\sin \frac{\varphi}{2} e_{1}+\cos \frac{\varphi}{2} e_{2}\right)\right),
\end{aligned}
$$

where

$$
\boldsymbol{e}_{1}:=\frac{1}{2} \sec \frac{\theta}{2}\left(f_{u}+f_{v}\right), \quad e_{2}:=\frac{1}{2} \csc \frac{\theta}{2}\left(-f_{u}+f_{v}\right)
$$

Then $\hat{\nu}$ is a unit normal vector field of $\hat{f}: D \rightarrow \mathbb{R}^{3}$, and the first and second fundamtal forms are

$$
\begin{aligned}
d \hat{f} \cdot d \hat{f} & :=d u^{2}+2 \cos \varphi d u d v+d v^{2} \\
\widehat{I I} & :=2 \sin \varphi d u d v
\end{aligned}
$$

That is, $\hat{f}$ is a Bäcklund transformation of $f$, and the asymptotic Chebyshev net of $\hat{f}$ coincides with that of $f$.

## Exercises

7-1 ${ }^{\mathrm{H}}$ Describe how Dini's pseudosphere

$$
f_{a, b}(u, v):=\left(\frac{a \cos u}{\cosh v}, \frac{a \sin u}{\cosh v}, a(v-\tanh v)+b u\right)
$$

obtained as a Bäcklund transformation of the line

$$
f_{0}(u, v)=(0,0, u+v)
$$

with unit normal vector field

$$
\nu_{0}(u, v)=(-\sin (u-v), \cos (u-v), 0) .
$$


[^0]:    27. May, 2016.
