1 A review of the surface theory.

The Euclidean space. We denote by $\mathbb{R}^3$ the *Euclidean* 3-*space* with *inner product* " $\cdot$ ", that is,

$$\boldsymbol{x}\cdot\boldsymbol{y} := x_1y_1+x_2y_2+x_3y_3 = {}^t\boldsymbol{x}\boldsymbol{y}, \qquad \left(\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right).$$

Here, we consider vectors in $\mathbb{R}^3$ as column vectors and "^{t} " denotes the *transposition*. The *Euclidean distance* d of $\mathbb{R}^3$ is defined as $d(\boldsymbol{x},\boldsymbol{y}) := |\boldsymbol{y}-\boldsymbol{x}|$, where $|\boldsymbol{v}| := \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$. An *isometry* of $\mathbb{R}^3$, that is, a transformation $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ preserving the Euclidean distance, has the following form:

$$(1.1) \qquad F(\boldsymbol{x}) = P\boldsymbol{x} + \boldsymbol{b} \qquad P \in \mathrm{O}(3), \quad \boldsymbol{b} \in \mathbb{R}^3.$$

Here, we denote

$$\begin{aligned} \mathrm{O}(3) &= \text{the set of } 3\times 3 \text{ orthogonal matrices}, \\ \mathrm{SO}(3) &= \{P \in \mathrm{O}(3) \mid \det P = 1\}. \end{aligned}$$

A basis $\{\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3\}$ of $\mathbb{R}^3$ is said to be *positive* (resp. *negative*) if $\det(\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3)$ is positive (resp. negative). The triple of the column vectors of a matrix in $\mathrm{O}(3)$ (resp. $\mathrm{SO}(3)$) forms an *orthonormal basis* (resp. a *positive orthonormal basis*).

An isometry as in (1.1) is called an *orientation preserving isometry* (resp. an *orientation reversing isometry*) if $A \in \mathrm{SO}(3)$ (resp. $A \in \mathrm{O}(3)\setminus\mathrm{SO}(3)$).

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Immersed surfaces. Consider a smooth[1] 2-*manifold* Σ and a smooth map

$$(1.2) \qquad f\colon \Sigma \ni \mathrm{P} \longmapsto f(\mathrm{P}) = {}^t\big(x(\mathrm{P}), y(\mathrm{P}), z(\mathrm{P})\big) \in \mathbb{R}^3.$$

Then each component of f is a smooth function defined on Σ. For each point $\mathrm{P}\in\Sigma$, we define a liner map $df_{\mathrm{P}}\colon T_{\mathrm{P}}\Sigma \to \mathbb{R}^3$ as

$$(1.3) \qquad df_{\mathrm{P}}(X) := {}^t\big(dx_{\mathrm{P}}(X), dy_{\mathrm{P}}(X), dz_{\mathrm{P}}(X)\big) \quad (X \in T_{\mathrm{P}}\Sigma),$$

which is called the *differential* of the map f at P, where dx, dy and dz are the usual differential of smooth functions.

Definition 1.1. A map f as in (1.2) is immersive at P if the map $df_{\mathrm{P}}\colon T_{\mathrm{P}}\Sigma \to \mathbb{R}^3$ is injective. Moreover, f is said to be an *immersion* if f is immersive at all $\mathrm{P}\in\Sigma$. In this lecture, an immersion as (1.2) is called an *immersed surface*.

Let $(U,(u,v))$ be a local coordinate chart of Σ at P. Then f in (1.2) is considered as an $\mathbb{R}^3$-valued function of (u,v), and

$$df\left(\frac{\partial}{\partial u}\right) = f_u = {}^t(x_u, y_u, z_u), \qquad df\left(\frac{\partial}{\partial v}\right) = f_v.$$

In particular, the image of $df_{(u,v)}$ is spanned by $f_u(u,v)$, $f_v(u,v)$. Thus, we have

Proposition 1.2. *The map $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ is an immersion if and only if $f_u(u,v)$ and $f_v(u,v)$ are linearly independent for each $(u,v)\in U$.*

[1]We use the word smooth for "of class C^∞", in this lecture.

Change of Parameters. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of a 2-manifold Σ. Take local coordinate charts $(U,(u,v))$ and $(V,(\xi,\eta))$ on a neighborhood of $\mathrm{P} \in \Sigma$. Then the change of coordinates is a pair of smooth functions

$$u = u(\xi,\eta), \qquad v = v(\xi,\eta) \tag{1.4}$$

such that (ξ,η) in V and $\bigl(u(\xi,\eta), v(\xi,\eta)\bigr) \in U$ corresponds to the same point of Σ. We write its *Jacobian matrix*

$$J := \begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix}, \tag{1.5}$$

such that $\det J$ does not vanish everywhere. We can write

$$du = u_\xi d\xi + u_\eta d\eta, \qquad dv = v_\xi d\xi + v_\eta d\eta. \tag{1.6}$$

The coordinate change (1.4) is said to be *orientation preserving* (resp. *orientation reversing*) if $\det J$ is positive (resp. negative). Two coordinate charts $(U,(u,v))$ and $(V,(\xi,\eta))$ are *compatible* if the change of coordinates is orientation preserving.

Definition 1.3. A manifold Σ is *orientable* if there exists an atlas $\mathcal{A} = \{(U_j,(u_j,v_j))\}$ of Σ such that each charts in $\mathcal{A}$ are compatible. Such a choice of atlas is called the *orientation* of Σ. The manifold Σ is called *oriented* if one orientation is specified. In this case, a coordinate chart $(U,(u,v))$ is said to be *compatible to the orientation* if it is compatible to one of the chart in the fixed orientation.

For the sake of simplicity, we consider only oriented manifolds in this lecture.

The unit normal vector.

Definition 1.4. The *unit normal vector field* ν of an immersion $f\colon \Sigma \to \mathbb{R}^3$ on a domain $U \subset \Sigma$ is a smooth map $\nu\colon \Sigma \supset U \to \mathbb{R}^3$ such that

$$df_{\mathrm{P}}(T_{\mathrm{P}}\Sigma) \perp \nu(\mathrm{P}), \qquad |\nu(\mathrm{P})| = 1 \tag{1.7}$$

hold for all $\mathrm{P} \in U$.

Remark 1.5. For a local coordinate chart $(U,(u,v))$ of Σ,

$$\nu(u,v) := \frac{f_u(u,v) \times f_v(u,v)}{|f_u(u,v) \times f_v(u,v)|} \tag{1.8}$$

is a unit normal vector field on U of $f\colon \Sigma \to \mathbb{R}^3$, where "$\times$" denotes the *vector product* or the *outer product* of $\mathbb{R}^3$.

Since (1.8) does not depend on the *orientation preserving* change of coordinates, one can find globally defined unit normal vector field of f if Σ is oriented.

The first fundamental form Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion and $(U,(u,v))$ a coordinate chart of Σ. The *first fundamental form* (or the *induced metric*) of f is defined as

$$\begin{aligned} ds^2 &:= df \cdot df = (f_u\,du + f_v\,dv)\cdot(f_u\,du + f_v\,dv) \\ &= E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ &\qquad E = f_u\cdot f_u, \quad F = f_u \cdot f_v, \quad G = f_v \cdot f_v. \end{aligned} \tag{1.9}$$

The functions E, F, G in (u,v) are called the *entries* of the first fundamental form. Let $(V,(\xi,\eta))$ be another coordinate chart of Σ, and consider a change of coordinates as in (1.4). Then we have, by the chain-rule,

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = {}^tJ \begin{pmatrix} E & F \\ F & G \end{pmatrix} J, \qquad (1.10)$$
$$\text{where} \quad \widetilde{E} = f_\xi \cdot f_\xi, \quad \widetilde{F} = f_\xi \cdot f_\eta, \quad \widetilde{G} = f_\eta \cdot f_\eta,$$

where J is the Jacobian matrix (1.5) of the coordinate change. Moreover, the first fundamental form does not depend of the choice of coordinate charts by virtue of (1.6):

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 = \widetilde{E}\,d\xi^2 + 2\widetilde{F}d\xi, d\eta + \widetilde{G}\,d\eta^2.$$

By the Schwartz inequality, we have

Lemma 1.6. $EG - F^2 > 0$.

Lemma 1.7. *Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion and $(U,(u,v))$ a local coordinate chart. Then*

$$\boldsymbol{e}_1(u,v) := \frac{1}{\sqrt{E}} f_u, \quad \boldsymbol{e}_2(u,v) := \frac{1}{\sqrt{EG - F^2}}\bigl(-F f_u + E f_v\bigr)$$

form an orthonormal system for each $(u,v) \in U$.

The second fundamental form. Let Σ be an oriented manifold, $f\colon \Sigma \to \mathbb{R}^3$ an immersion, and ν the unit normal vector field. The *second fundamental form* is defined as

$$II := -df \cdot d\nu = -(f_u\,du + f_v\,dv) \cdot (\nu_u\,du + \nu_v\,dv), \qquad (1.11)$$

where $(U,(u,v))$ is any local coordinate chart of Σ.

Lemma 1.8. *The second fundamental form is written as*

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$
$$\textit{where} \quad \begin{cases} L = -f_u \cdot \nu_u = f_{uu} \cdot \nu, \\ M = -f_u \cdot \nu_v = -f_v \cdot \nu_u = f_{uv} \cdot \nu, \\ N = -f_v \cdot \nu_v = f_{vv} \cdot \nu. \end{cases}$$

Proof. By definition,

$$II = -f_u \cdot \nu_u\,du^2 - (f_u \cdot \nu_v + f_v \cdot \nu_u)\,du\,dv - f_v \cdot \nu_v\,dv^2$$

holds. Here, since f_u and f_v are perpendicular to ν,

$$-f_u \cdot \nu_v = -(f_u \cdot \nu)_v + f_{uv} \cdot \nu = f_{uv} \cdot \nu,$$
$$-f_v \cdot \nu_u = -(f_v \cdot \nu)_u + f_{vu} \cdot \nu = -f_{uv} \cdot \nu, \dots \qquad \square$$

The functions L, M, N in Lemma 1.8 is called the *entries* of the second fundamental form.

Similar to the first fundamental form, the second fundamental form does not depend of the choice of coordinates:

$$\begin{pmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} = {}^tJ \begin{pmatrix} L & M \\ M & N \end{pmatrix} J, \qquad (1.12)$$

where L, M, N (resp. $\widetilde{L}$, $\widetilde{M}$, $\widetilde{N}$) are the entries of the second fundamental form in uv- (resp. $\xi\eta$-) coordinates, and J is the Jacobian matrix (1.5).

Curvatures. Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of an oriented 2-manifold Σ and take the unit normal vector field ν by Remark 1.5. For a local coordinate chart $(U,(u,v))$ on Σ, one can consider a matrix-valued function by Lemma 1.6:

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}, \tag{1.13}$$

which is called the *Weingarten matrix*. Taking another coordinate chart $(V,(\xi,\eta))$, one can obtain the Weingarten matrix $\widetilde{A}$ with respect to the coordinates (ξ,η). By (1.10) and (1.12)

$$\widetilde{A} = J^{-1}AJ \tag{1.14}$$

holds, where J is the Jacobian matrix as in (1.5). Hence

Lemma 1.9. *The eigenvalues, the determinant, and the trace of the Weingarten matrix* (1.13) *do not depend on the choice of coordinates* (*compatible to the orientation*).

Lemma 1.10. *The eigenvalues of the Weingarten matrix are invariant under orientation preserving isometries.*

Lemma 1.11 (Theorem 8.7 in [1-1])**.** *The eigenvalues of the Weingarten matrix are real valued functions.*

Definition 1.12. The eigenvalues λ_1, λ_2 of the Weingarten matrix A is called the *prinicipal curvatures*. We call the functions

$$K := \det A = \lambda_1\lambda_2, \qquad H := \frac{1}{2}\operatorname{tr} A = \frac{\lambda_1+\lambda_2}{2}, \tag{1.15}$$

the *Gaussian curvature* and the *mean curvature*, respectively.

Example 1.13. An immersion $(u,v) \mapsto (u,v,0)$ represents the xy-plane in $\mathbb{R}^3$. Since $\nu = {}^t(0,0,1)$ is constant, the Weingarten matrix vanishes identically, and then the principal curvatures are zero. In particular, the plane has zero Gaussian curvature.

Example 1.14. An map $f(u,v) = {}^t(\cos u\cos v, \cos u \sin v, \sin u)$ is immersive on $(-\frac{\pi}{2},\frac{\pi}{2}) \times (-\pi,\pi)$, which represents the sphere of radius 1, whose Gaussian curvature is identically 1.

References

[1-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

Exercises

1-1[H] Consider a smooth map $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ as

$$f(u,v) = \left(\frac{a\cos u}{\cosh v}, \frac{a \sin u}{\cosh v}, a(v-\tanh v)+bu\right),$$

where $a>0$ and $b \geqq 0$ are constants satisfying $a^2+b^2=1$.

(1) Find a domain $D \subset \mathbb{R}^2$ satisfying
- The restriction $f|_D$ is an immersion.
- $(0,1) \in D$.
- D is maximal among domains satisfying two conditions above.

(2) Compute the Gaussian curvature of f on D.

(3) Draw a picture of the image of $f|_D$ for $(a,b)=(1,0)$.

2 The Gauss and Weingarten formulae

In this section, we consider an immersion $f\colon \mathbb{R}^2 \supset U \to \mathbb{R}^3$ of the domain U in the uv-plane, and let ν be the unit normal vector field as

$$\nu := \frac{f_u \times f_v}{|f_u \times f_v|}.$$

Then, for each $\mathrm{P} = (u, v) \in U$,

$$\mathcal{F}(u,v) := \{f_u(u,v), f_v(u,v), \nu(u,v)\} \tag{2.1}$$

forms a positive basis of $\mathbb{R}^3$. In this lecture, we call $\mathcal{F}$ the *Gauss frame* of f. In particular, 2-dimensional vector space spanned by $\{f_u(u,v), f_v(u,v)\}$ is the image of $T_{\mathrm{P}}U$ by the differential map:

$$\operatorname{Span}\{f_u(\mathrm{P}), f_v(\mathrm{P})\} = df(T_{\mathrm{P}}U).$$

We call this vector space by the *tangent vector space* of the surface at P. The tangent vector space is characterized as the orthogonal complement of $\nu(\mathrm{P})$, and we have the orthogonal decomposition

$$\mathbb{R}^3 = \left(T_{f(\mathrm{P})}\mathbb{R}^3\right) = df(T_{\mathrm{P}}U) \oplus \mathbb{R}\nu_{\mathrm{P}}. \tag{2.2}$$

From now on, we denote by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2$$

the first and the second fundamental forms, respectively.

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The Weingarten Formula. The following formula mesures the change of the unit normal vector in terms of the entries of the fundamental forms (cf. Lemma 8.5 in [2-1] (page 85)):

Theorem 2.1 (The Weingarten Formula)**.** *It holds that*

$$\begin{cases} \nu_u = -A_1^1 f_u - A_1^2 f_v, \\ \nu_v = -A_2^1 f_u - A_2^2 f_v, \end{cases}$$
$$\left(A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_1^2 & A_2^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \right).$$

The Christoffel symbols and the Gauss Formula.

Definition 2.2 (The Christoffel symbols, [2-1], page 108)**.** The following $\varGamma_{ij}^k$ $(i,j,k = 1,2)$ are called the *Christoffel symbols*:

$$\begin{cases} \varGamma_{11}^1 := \dfrac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \\ \varGamma_{11}^2 := \dfrac{2EF_u - EE_v - FE_u}{2(EG - F^2)}, \\ \varGamma_{12}^1 = \varGamma_{21}^1 := \dfrac{GE_v - FG_u}{2(EG - F^2)}, \\ \varGamma_{12}^2 = \varGamma_{21}^2 := \dfrac{EG_u - FE_v}{2(EG - F^2)}, \\ \varGamma_{22}^1 := \dfrac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \\ \varGamma_{22}^2 := \dfrac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}. \end{cases} \tag{2.3}$$

By straightforward calculations, we have

Lemma 2.3.

$$
\begin{aligned}
&E\Gamma^1_{11} + F\Gamma^2_{11} = \frac{1}{2}E_u, && F\Gamma^1_{11} + G\Gamma^2_{11} = F_u - \frac{1}{2}E_v\\
&E\Gamma^1_{12} + F\Gamma^2_{12} = \frac{1}{2}E_v, && F\Gamma^1_{12} + G\Gamma^2_{12} = \frac{1}{2}G_u,\\
&E\Gamma^1_{22} + F\Gamma^2_{22} = F_v - \frac{1}{2}G_u && F\Gamma^1_{22} + G\Gamma^2_{22} = \frac{1}{2}G_v,\\
&\Gamma^1_{11} + \Gamma^2_{12} = \frac{g_u}{2g}, && \Gamma^1_{21} + \Gamma^2_{22} = \frac{g_v}{2g},
\end{aligned}
$$

where $g := EG - F^2$.

The Gauss formula ([2-1], Proposition 11.1 in page 122) is satated using the Christoffel symbols as follows:

Theorem 2.4 (The Gauss Formula)**.** *Let* $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ *be an immersion and* ν *its unit normal vector field, and let* Γ^i_{jk} $(i,j,k = 1,2)$ *and* L*,* M*,* N *be the Christoffel symbols and the entries of the second fundamental forms, respectively. Then it holds taht*

$$
\begin{cases}
f_{uu} = \Gamma^1_{11}f_u + \Gamma^2_{11}f_v + L\nu,\\
f_{uv} = \Gamma^1_{12}f_u + \Gamma^2_{12}f_v + M\nu,\\
f_{vv} = \Gamma^1_{12}f_u + \Gamma^2_{22}f_v + N\nu.
\end{cases}
\tag{2.4}
$$

The Gauss Frame and the Fundamental Equations. Combining Theorems 2.1 and 2.4, we have the following *Fundamental Equations for surface theory*:

Theorem 2.5. *Let* $f\colon U \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ *be an immersion of a domain* U *in the* uv*-plane. Then the Gauss frame* $\mathcal{F} := \{f_u, f_v, \nu\}$ *as in* (2.1) *satisfies the equations*

$$
\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda
\tag{2.5}
$$

$$
\Omega := \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & -A^1_1\\ \Gamma^2_{11} & \Gamma^2_{12} & -A^2_1\\ L & M & 0 \end{pmatrix}, \qquad \Lambda := \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} & -A^1_2\\ \Gamma^2_{21} & \Gamma^2_{22} & -A^2_2\\ M & N & 0 \end{pmatrix},
$$

where Γ^i_{jk} $(i,j,k = 1,2)$*,* A^k_l *and* L*,* M*,* N *are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.*

Since the coefficient matrices Ω, Λ in (2.5) are expressed in terms of the first and the second fundamenatal forms, we have the followoing, which is the "uniquness part" of the fundamental theorem for surface theory (which will be stated in Section 4):

Corollary 2.6. *Let* f*,* $\tilde{f}\colon U \to \mathbb{R}^3$ *be two immersions of a domain* $U \subset \mathbb{R}^2$*. If the first and the second fundamental forms of* f *and* $\tilde{f}$ *are common, there exist a matrix* $P \in \mathrm{SO}(3)$ *and a vector* $\boldsymbol{p} \in \mathbb{R}^3$ *such that*

$$
\tilde{f} = Pf + \boldsymbol{p}.
\tag{2.6}
$$

In other words, the first and the second fundamental forms determines a surface uniquely up to (*orientation preserving*) *isometries of* $\mathbb{R}^3$*.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}_1$ be the Gauss frames of f and $\tilde{f}$, respectivelymtively. By Theorem 2.5, they astisfy the equation (2.5) with common coefficient matrices Ω and Λ. Hence

$$\frac{\partial}{\partial u}\mathcal{F}\mathcal{F}_1^{-1} = \mathcal{F}_{1,u}\mathcal{F}^{-1} + \mathcal{F}_1\left(\mathcal{F}^{-1}\right)_u = \mathcal{F}_{1,u}\mathcal{F}^{-1} - \mathcal{F}_1\left(\mathcal{F}^{-1}\mathcal{F}_u\mathcal{F}^{-1}\right)$$
$$= \mathcal{F}_1\Omega\mathcal{F}^{-1} - \mathcal{F}_1\Omega\mathcal{F}_1{}^{-1} = O,$$
$$\frac{\partial}{\partial v}\mathcal{F}_1\mathcal{F}^{-1} = O.$$

Since the domain D is connected, it follows that $\mathcal{F}_1\mathcal{F}^{-1} = P$, for some constant matrix P, that is, $\mathcal{F}_1 = P\mathcal{F}$ holds. Hence

$$\tilde{f}_u = Pf_u, \qquad \tilde{f}_v = Pf_v, \qquad \tilde{\nu} = P\nu$$

hold, where $\tilde{\nu}$ is the unit normal vector field of $\tilde{f}$. This implies

$$(\tilde{f} - Pf)_u = (\tilde{f} - Pf)_v = \mathbf{0},$$

and hence $\tilde{f} - Pf =: \boldsymbol{p}$ is a constant vector, which yields (2.6). So, it is sufficient to show that $P \in \mathrm{SO}(3)$. Fix P $:= (u_0, v_0) \in U$ and set

$$\boldsymbol{f}_1 = f_u(u_0, v_0), \qquad \boldsymbol{f}_2 = f_v(u_0, v_0), \qquad \boldsymbol{f}_3 = \nu(u_0, v_0).$$

Then

$$\boldsymbol{f}_1 \cdot \boldsymbol{f}_1 = E(u_0, v_0) = \tilde{f}_u(u_0, v_0) \cdot \tilde{f}_u(u_0, v_0) = P\boldsymbol{f}_1 \cdot P\boldsymbol{f}_1,$$

and similarly, we have

$$\boldsymbol{f}_j \cdot \boldsymbol{f}_k = P\boldsymbol{f}_j \cdot P\boldsymbol{f}_k \qquad (j, k = 1, 2, 3).$$

Since $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}$ forms a basis of $\mathbb{R}^3$, we can conclude that

$$\boldsymbol{x} \cdot \boldsymbol{y} = P\boldsymbol{x} \cdot P\boldsymbol{y} \qquad (\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3),$$

and hence P is a orthogonal matrix. Moreover, both $\det \mathcal{F}$ and $\det \widetilde{\mathcal{F}}$ are positive since $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are positively oriented. Thus, $\det P > 0$, that is, $P \in \mathrm{SO}(3)$.

□

Theorema egregium Differentiate the first (resp. the second) equation of (2.4) in v and u, respectively, we obtain

$$\begin{aligned} f_{uuv} &= (*) + (\Gamma^2_{11,v} + \Gamma^1_{11}\Gamma^2_{11} + \Gamma^2_{11}\Gamma^2_{12} - LA^2_2)f_v, \\ f_{uvu} &= (*) + (\Gamma^2_{12,u} + \Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^2_{22} - MA^2_1)f_v, \end{aligned} \tag{2.7}$$

here $(*)$'s are linear cobinations of f_u and ν. Since $f_{uuv} = f_{uvu}$, we have

$$\begin{aligned} K = {} & \frac{E\left(E_vG_v - 2F_uG_v + G_u{}^2\right)}{4(EG - F^2)^2} \\ & + \frac{F(E_uG_v - E_vG_u - 2E_vF_v - 2F_uG_u + 4F_uF_v)}{4(EG - F^2)^2} \\ & + \frac{G\left(E_uG_u - 2E_uF_v + E_v{}^2\right)}{4(EG - F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG - F^2)} \end{aligned} \tag{2.8}$$

comparing the coefficients of f_v in (2.7) and substituting (2.3), here K is the Gaussian curvature defined in (1.15). The equality (2.8) is known as Gauss' *Theorema Egregium* ("remarkable theorem"), cf. Theorem 11.2 of [2-1].

References

[2-1] 梅原雅顕・山田光太郎：曲線と曲面—微分幾何的アプローチ（改訂版），裳華房，2014.

[2-2] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976.

Exercises

2-1[H] Assume $f : U \ni (u, v) \mapsto f(u, v) \in \mathbb{R}^3$ be an immersion of a domain U in $\mathbb{R}^2$, whose first and second fundamental forms are expressed as

$$ds^2 = du^2 + 2\cos\theta \, du \, dv + dv^2, \qquad II = 2\sin\theta \, du \, dv,$$

where $\theta(u, v)$ is a smooth function in (u, v).

(1) Find the condition of θ for f to be an immersion.

(2) Compute the Christoffel symbols.

(3) Write down the equation (2.8) in terms of θ.

3 The Gauss and Codazzi equations

The compatibility conditions. As seen in Theorem 2.5 in Section 2, the Gauss frame $\mathcal{F} = (f_u, f_v, \nu)$ for an immersion $f\colon D \ni (u,v) \mapsto f(u,v) \in \mathbb{R}^3$ satisfies the equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\Lambda \tag{3.1}$$

$$\Omega := \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -A_1^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -A_1^2 \\ L & M & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \Gamma_{21}^1 & \Gamma_{22}^1 & -A_2^1 \\ \Gamma_{21}^2 & \Gamma_{22}^2 & -A_2^2 \\ M & N & 0 \end{pmatrix},$$

where Γ_{jk}^i $(i,j,k = 1,2)$, A_l^k and L, M, N are the Christoffel symbols, the entries of the Weingarten matrix and the entries of the second fundamental form, respectively.

Lemma 3.1. *The coefficient matrices Ω, Λ in* (3.1) *satisfy*

$$\frac{\partial \Omega}{\partial v} - \frac{\partial \Lambda}{\partial u} = \Omega\Lambda - \Lambda\Omega. \tag{3.2}$$

Proof. Differentiating the first equation in (3.1) with respect to v, we have

$$\mathcal{F}_{uv} = (\mathcal{F}\Omega)_v = \mathcal{F}_v\Omega + \mathcal{F}\Omega_v = \mathcal{F}\Lambda\Omega + \mathcal{F}\Omega_v = \mathcal{F}(\Lambda\Omega + \Omega_v).$$

Similarly, differentiating the first equation in (3.1) in u, it holds that

$$\mathcal{F}_{vu} = \mathcal{F}(\Omega\Lambda + \Lambda_u).$$

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Thus,

$$\mathcal{F}(\Lambda\Omega + \Omega_v) = \mathcal{F}(\Omega\Lambda + \Lambda_u)$$

holds. Noticing that $\mathcal{F}$ is a regular matrix, we have the conclusion. □

The equation (3.2) is called the *compatibility condition*, or the *integrability condition* of the equation (3.1).

The Gauss and Codazzi equations. The compatibility condition (3.2) consists of nine equations, because it is the equality for 3×3 matrices. However, they can be reduced three equations:

Lemma 3.2. *The compatibility condition* (3.2) *is equivalent to the equation (Equation* (2.8)*)*

$$\begin{aligned} K = {} & \frac{E\bigl(E_vG_v - 2F_uG_v + {G_u}^2\bigr)}{4(EG-F^2)^2} \\ & + \frac{F(E_uG_v - E_vG_u - 2E_vF_v - 2F_uG_u + 4F_uF_v)}{4(EG-F^2)^2} \\ & + \frac{G\bigl(E_uG_u - 2E_uF_v + {E_v}^2\bigr)}{4(EG-F^2)^2} - \frac{E_{vv} - 2F_{uv} + G_{uu}}{2(EG-F^2)} \end{aligned} \tag{3.3}$$

and the following two equations:

$$\begin{aligned} L_v - M_u &= \Gamma_{21}^1 L + \Gamma_{21}^2 M - \Gamma_{11}^1 M - \Gamma_{11}^2 N, \\ M_v - N_u &= \Gamma_{22}^1 L + \Gamma_{22}^2 M - \Gamma_{12}^1 M - \Gamma_{12}^2 N. \end{aligned} \tag{3.4}$$

Proof. By a direct computations, we can conclude that the $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$-components of (3.2) are equivalent to (2.8). On the other hand, the first (resp. the second) equation in (3.4) is equivalent to the $(3,1)$ (resp. $(3,2)$) component of (3.2). Moreover, the $(1,3)$ and $(2,3)$-components are equivalent to (3.4) because of the definition of the Weingarten matrix

$$A = \frac{1}{g}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \qquad (g = EG - F^2)$$

and Lemma 2.3. □

The equation (2.8) is called the *Gauss equation*. On the other hand, the equations (3.4) are called the *Codazzi equations*, or the *Codazzi-Mainardi equations*.

Corollary 3.3. *Let $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ be an immersion with first and second fundamental forms as*

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \quad II = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

Then the entries E, F, G, L, M and N satisfy the Gauss equation (3.3) *and the Codazzi equation* (3.4)*, where Γ^i_{jk}'s are the Christoffel symbols* (2.3)*, and K is the Gaussian curvature in* (1.15).

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Exercises

3-1[H] Assume that the first and second fundamental forms of the immersion $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ are

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2, \qquad II = 2M\,du\,dv.$$

(1) Show that the Codazzi equations are

$$M_u + \left(2\Gamma^2_{12} - \frac{g_u}{2g}\right) M = 0, \quad M_v + \left(2\Gamma^1_{12} - \frac{g_v}{2g}\right) M = 0,$$

where $g = EG - F^2$.

(2) Moreover, if the Gaussian curvature is negative constant, show that $E_v = 0$ and $G_u = 0$ hold.

3-2 Let f be an immersion of the uv-plane into $\mathbb{R}^3$. The parameter (u, v) is said to be *isothermal* or *conformal* if the first fundamental form is written as

$$ds^2 = e^{2\sigma}(du^2 + dv^2) \qquad \text{(i.e. } E = G = e^{2\sigma}, F = 0),$$

where $\sigma = \sigma(u, v)$ is a smooth function in (u, v). [2]

Assume that f is parametrized by an isothermal parameter (u, v).

(1) Show that the Gauss frame $\mathcal{F}$ satisfies the equation

$$\frac{\partial \mathcal{F}}{\partial u} = \mathcal{F}\varOmega, \qquad \frac{\partial \mathcal{F}}{\partial v} = \mathcal{F}\varLambda \tag{3.5}$$

$$\varOmega := \begin{pmatrix} \sigma_u & \sigma_v & -e^{-2\sigma}L \\ -\sigma_v & \sigma_u & -e^{-2\sigma}M \\ L & M & 0 \end{pmatrix},$$

$$\varLambda := \begin{pmatrix} \sigma_v & -\sigma_u & -e^{-2\sigma}M \\ \sigma_u & \sigma_v & -e^{-2\sigma}N \\ M & N & 0 \end{pmatrix},$$

where L, M and N are the entries of the second fundamental form.

[2] Let (Σ, ds^2) be an arbitrary 2-dimensional Riemannian manifold. Then, it is known that, for any point $\mathrm{P} \in S$, there exists an isothermal coordinate chart (u, v) containing P, that is, the Riemannian metric ds^2 is written as $ds^2 = e^{2\sigma}(du^2 + dv^2)$ (cf. Section 15 of [3-1]).

(2) Verify that the Gauss and Codazzi equations are written as

$$\begin{aligned} \sigma_{uu} + \sigma_{vv} + e^{-2\sigma}(LN - M^2) &= 0 \\ L_v - M_u &= \sigma_v(L + N) \\ N_u - M_v &= \sigma_u(L + N). \end{aligned}$$

3-3 Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of an oriented 2-manifold Σ, and (u, v) be an isothermal coordinate system around $\mathrm{P} \in \Sigma$ compatible to the orientation of Σ, and (ξ, η) be another coordinate system around P compatible to the orientation of Σ.

(1) Show that (ξ, η) is isothermal if and only if

$$u_\xi = v_\eta, \qquad u_\eta = -v_\xi.$$

(2) Verify that the above conditions are equivalent to that

$$\zeta := \xi + i\eta \mapsto z := u + iv$$

is holomoprhic. [3]

[3] Hence, the existence of isothermal coordinates implies the existence of the structure of a Riemann surface (a 1-dimensional complex manifold) on an oriented Riemannian manofold.

3-4 Let $f\colon D \to \mathbb{R}^3$ be an immersion with an isothremal parameter (u,v), with fundamental forms

$$ds^2 = e^{2\sigma}(du^2 + dv^2), \quad I\!I = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

(1) Show that the Gauss and Codazzi equations are equivalent to

$$-e^{-2\sigma}(\sigma_{uu} + \sigma_{vv}) = K, \qquad \frac{1}{2}\frac{\partial H}{\partial z} = e^{-2\sigma}\frac{\partial q}{\partial \bar{z}},$$

where $z = u + iv$ be an complex coordinate,

$$q := \frac{1}{4}\bigl((L-N) - 2iM\bigr),$$

K is the Gaussian curvature, H is the mean curvature, and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

(2) When H is constant, verify that the Codazzi equation is equivalent to the holomorphicity of q.

4 The fundamental theorem for surfaces

We shall give a proof of the following theorem in this section (cf. Appendix B-10 in [4-1]):

Theorem 4.1 (The fundamental theorem for surface theory). *Let D be a simply connected domain of $\mathbb{R}^2$ and let $E(>0)$, F, $G(>0)$, L, M and N be a C^∞-functions on D satisfying $EG-F^2>0$, the Gauss equation* (3.3)*, and the Codazzi equations* (3.4)*. Then there exists an immersion $f\colon D\to\mathbb{R}^3$ whose first and second fundamental forms are*

$$ds^2 = E\,du^2+2F\,du\,dv+G\,dv^2,\quad I\!I = L\,du^2+2M\,du\,dv+N\,dv^2.$$

Moreover, such an immersion f is unique up to rotations and parallel translations.

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Theorem 4.2 (The fundamental theorem). *Let V be a finite dimensional vector space over $\mathbb{R}$ and denote by $\operatorname{Hom}(V)$ the space of linear transformations on V. Take a C^∞-map $A\colon I\to\operatorname{Hom}V$ defined on an interval $I\subset\mathbb{R}$. Then for arbitrary t_0 and $\boldsymbol{v}_0\in V$, there exists a unique C^∞-map $\boldsymbol{v}\colon I\to V$ satisfying*

$$\text{(4.1)}\qquad \frac{d\boldsymbol{v}}{dt}(t)=A(t)\boldsymbol{v}(t),\qquad \boldsymbol{v}(t_0)=\boldsymbol{v}_0.$$

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The equation (4.1) is called an *initial value problem of a linear differential equation.*[4] We denote the unique solution of (4.1) by $\boldsymbol{v}_{A,t_0,\boldsymbol{v_0}}$.

Theorem 4.3. *Under the same notations as in Theorem 4.2, let $A\colon I\times U\to\operatorname{Hom}(V)$ and and $\boldsymbol{v}_0\colon I'\to V$ be C^∞-maps where I, I' are intervals and $U\subset\mathbb{R}^n$ is a domain. Then for arbitrarily fixed $t_0\in I$,*

$$\mathbb{R}^3\supset I\times U\times I'\ni(t,\boldsymbol{\alpha},\beta)\longmapsto \boldsymbol{v}_{A(*,\boldsymbol{\alpha}),t_0,\boldsymbol{v}_0(\beta)}\in V$$

is a C^∞-map.

Theorem 4.3 is called the *regularity of the solutions of ordinary differential equations with respect to parameters and initial conditions.*

From now on we denote by $\mathrm{M}(n,\mathbb{R})$ (resp. $\mathrm{GL}(n,\mathbb{R})$) the vector space consists of the $n\times n$-real matrices (resp. the $n\times n$-regular matrices).

Corollary 4.4. *Let $\Omega\colon I\to\mathrm{M}(n,\mathbb{R})$ be a C^∞-map defined on an interval I. Then for $t_0\in I$ and an arbitrary matrix $A_0\in\mathrm{M}(n,\mathbb{R})$, there exists a unique C^∞-map $\mathcal{F}_{A_0}\colon I\to\mathrm{M}(n,\mathbb{R})$ satisfying*

$$\text{(4.2)}\qquad \frac{d\mathcal{F}}{dt}(t)=\mathcal{F}(t)\Omega(t),\qquad \mathcal{F}(t_0)=A_0.$$

Moreover,

[4] Compare with the well-known Cauchy's existence theorem. The solution of the linear differential equation is defined on the whole interval I where the coefficient A is defined. See [4-2] and [4-3].

- *if* $A_0 \in \mathrm{GL}(n,\mathbb{R})$ *then* $\mathcal{F}(t) \in \mathrm{GL}(n,\mathbb{R})$*, for* $t \in I$*,*
- $\mathcal{F}_B = B\mathcal{F}_{\mathrm{id}}$*, where* id *is the* $n \times n$*-identity matrix and* $\mathcal{F}_B$ *(resp.* $\mathcal{F}_{\mathrm{id}}$*) is the solution of* (4.2) *with* $A_0 = B$ ($A_0 = \mathrm{id}$).

Proof. The first part is a direct conclusion of Theorem 4.2 for $V = \mathrm{M}(n,\mathbb{R})$ and $A(t)\colon V \ni F \mapsto \Omega(t)F \in V$.

Let $\mathcal{F}$ be the solution of (4.2). Then it holds that,

$$\frac{d}{dt}\det\mathcal{F} = \operatorname{tr}\left(\widetilde{\mathcal{F}}\frac{d\mathcal{F}}{dt}\right) = \operatorname{tr}(\widetilde{\mathcal{F}}\mathcal{F}\Omega) = \det\mathcal{F}\operatorname{tr}(\Omega),$$

where $\widetilde{\mathcal{F}}$ is the cofactor matrix of $\mathcal{F}$. Then $f := \det\mathcal{F}$ satisfies

$$\frac{df}{dt} = f\omega, \quad f(t_0) = a_0, \quad \text{where } \omega = \operatorname{tr}\Omega \text{ and } a_0 = \det A_0.$$

The unique solution of above equation is

$$f(t) = a_0 \exp\left(\int_{t_0}^{t}\omega(s)\,ds\right),$$

which never vanish if $a_0 \neq 0$. Final assertion holds by the uniqueness of the solution of (4.2). □

Integrable Partial Differential Equations. Let D be a domain in the uv-plane $\mathbb{R}^2$ and take C^∞ maps $\Omega, \Lambda\colon D \to \mathrm{M}(n,\mathbb{R})$. In this section we consider a system of differential equations of unknown $\mathcal{F}\colon D \to \mathrm{M}(n,\mathbb{R})$:

$$\frac{\partial\mathcal{F}}{\partial u} = \mathcal{F}\Omega, \qquad \frac{\partial\mathcal{F}}{\partial v} = \mathcal{F}\Lambda, \quad \mathcal{F}(\mathrm{P}) = F_0 \in \mathrm{GL}(n,\mathbb{R}), \tag{4.3}$$

where $\mathrm{P} \in D$ is a fixed point.

Lemma 4.5. *Assume that there exists a solution* $\mathcal{F}$ *of* (4.3)*. Then* $\mathcal{F}(u,v) \in \mathrm{GL}(n,\mathbb{R})$ *for any* $(u,v) \in D$ *and it holds that*

$$\Omega_v - \Lambda_u = \Omega\Lambda - \Lambda\Omega. \tag{4.4}$$

Proof. Fix $\mathrm{Q} \in D$ and take a smooth path $\gamma(t) = \bigl(u(t), v(t)\bigr)$ ($0 \leqq t \leqq 1$) on D joining P and Q. Then $\mathcal{F}\circ\gamma(t)\colon [0,1] \to \mathrm{M}(n,\mathbb{R})$ satisfies

$$\begin{gathered}\frac{d\mathcal{F}\circ\gamma}{dt}(t) = \mathcal{F}\circ\gamma(t)\hat{\Omega}(t), \quad \mathcal{F}\circ\gamma(0) = F_0 \in \mathrm{GL}(n,\mathbb{R}),\\ \hat{\Omega}(t) := \Omega\circ\gamma(t)\dot{u}(t) + \Lambda\circ\gamma(t)\dot{v}(t).\end{gathered} \tag{4.5}$$

Then Corollary 4.4 implies that $\mathcal{F}(\mathrm{Q}) \in \mathrm{GL}(n,\mathbb{R})$. Since Q is arbitrary, the first assertion holds.

The second assertion can be proven by the same way in the proof of Lemma 3.1. □

Theorem 4.6. *Let* D *be a* <u>*simply connected*</u> *domain in* $\mathbb{R}^2$*. Then there exists a unique solution* $\mathcal{F}\colon D \to \mathrm{M}(n,\mathbb{R})$ *of* (4.3) *if* Ω *and* Λ *satisfy* (4.4)*.*

Proof. First we shall prove the uniqueness: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the solutions of (4.3). Since the values of $\mathcal{F}_j$ are regular matrices (Lemma 4.5), we can set $\mathcal{G} := \mathcal{F}_1\mathcal{F}_2^{-1}$. Then by the similar computation in the proof of Corollary 2.6, we have $\mathcal{G}_u = \mathcal{G}_v = O$, and hence $\mathcal{G}$ is constant on D:

$$\mathcal{G}(\mathrm{P}) = \mathcal{F}_1(\mathrm{P})\mathcal{F}_2(\mathrm{P})^{-1} = F_0F_0^{-1} = \mathrm{id}.$$

Then we have $\mathcal{F}_1 = \mathcal{F}_2$.

Next, we prove the existence. Take $\mathrm{Q} \in D$ arbitrarily and choose a path $\gamma(t) = \bigl(u(t), v(t)\bigr)$ $(0 \leqq t \leqq 1)$ joining P and Q, and consider the ordinary differential equation (4.5). Let $\mathcal{F}_\gamma \colon I \to \mathrm{GL}(n, \mathbb{R})$ be the unique solution (cf. Corollary 4.4) of (4.5), and set $\mathcal{F}(\gamma, \mathrm{Q}) := \mathcal{F}_\gamma(1)$.

We now prove that $\mathcal{F}$ does not depend on the choice of the path γ. Take another path $\tilde{\gamma}$ joining P and Q. Since D is simply connected, they are homotopically equivalent. In other words, we can take a smooth map $\sigma \colon [0,1] \times [0,1] \to D$ such that $\sigma(0,t) = \gamma(t)$, $\sigma(1,t) = \tilde{\gamma}(t)$, $\sigma(s,0) = \mathrm{P}$, $\sigma(s,1) = \mathrm{Q}$. We write $\sigma(s,t) = \bigl(u(s,t), v(s,t)\bigr)$ and set

$$S = \Omega \circ \sigma u_s + \Lambda \circ \sigma v_s, \quad T = \Omega \circ \sigma u_t + \Lambda \circ \sigma v_t.$$

Note that

$$S(s,1) = O \qquad (0 \leqq s \leqq 1), \tag{4.6}$$

because $\sigma(s,1)$ is constant. For each fixed $s \in [0,1]$, take the unique solution $\hat{\mathcal{F}}(s,t)$ of the ordinary differential equation

$$\frac{\partial \hat{\mathcal{F}}(s,t)}{\partial t} = \hat{\mathcal{F}}(s,t) T(s,t), \qquad \hat{\mathcal{F}}(s,0) = F_0. \tag{4.7}$$

Then by the regularity of the solution of ordinary differential equation with respect to the parameters, we have a smooth map $\hat{\mathcal{F}} \colon [0,1] \times [0,1] \to D$, and by definition,

$$F_0 = \hat{\mathcal{F}}(s,0), \quad \mathcal{F}(\gamma, \mathrm{Q}) = \hat{\mathcal{F}}(0,1), \quad \mathcal{F}(\tilde{\gamma}, \mathrm{Q}) = \hat{\mathcal{F}}(1,1),$$

that is, to show that $\mathcal{F}(\gamma, \mathrm{Q})$ does not depend on γ, it is sufficient to show that $\hat{\mathcal{F}}(0,1) = \hat{\mathcal{F}}(1,1)$. Noticing $S_t - T_s - ST + TS = O$

holds because of (4.4), we have

$$\begin{aligned}\bigl(\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S\bigr)_t &= \hat{\mathcal{F}}_{st} - \hat{\mathcal{F}}_t S - \hat{\mathcal{F}} S_t \\ &= \hat{\mathcal{F}}_{ts} - \mathcal{F} T S - \mathcal{F} S_t = (\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S) T.\end{aligned}$$

Hence for each fixed s, $\hat{\mathcal{F}}_s - \hat{F} S$ is another solution of the same equation (4.7) with the initial condition $\hat{\mathcal{F}}_s(s,0) - \hat{\mathcal{F}}(s,0) S(s,0) = O$. Hence $\hat{\mathcal{F}}_s - \hat{\mathcal{F}} S = O$ for $(s,t) \in [0,1] \times [0,1]$. In particular, $\hat{\mathcal{F}}_s(s,1) = \hat{\mathcal{F}}(s,1) S(s,1) = O$ and then $\hat{\mathcal{F}}(s,1)$ is constant.

Thus, by setting $\mathcal{F}(\mathrm{Q}) := \mathcal{F}(\gamma, \mathrm{Q})$, we have the map $\mathcal{F} \colon D \to \mathrm{M}(n, \mathbb{R})$. We finally prove that $\mathcal{F}$ satisfies the equation (4.3). Let $\mathrm{Q} = (u_0, v_0)$, $\mathrm{Q}_h = (u_0 + h, v_0)$ and set $\gamma(t) = (u_0 + th, v_0)$ $(t \in [0,1])$. Then $\mathcal{F}(\mathrm{Q}_h) = \hat{\mathcal{F}}(1)$, where $\hat{\mathcal{F}}$ is a solution of

$$\frac{d\hat{\mathcal{F}}}{dt} = h \hat{\mathcal{F}} \Omega \circ \gamma(t), \qquad \hat{\mathcal{F}}(0) = \mathcal{F}(\mathrm{Q}).$$

Thus, we can show

$$\mathcal{F}_u(\mathrm{Q}) = \lim_{h \to 0} \frac{\mathcal{F}(\mathrm{Q}_{-h}) - \mathcal{F}(\mathrm{Q})}{h} = \mathcal{F}(\mathrm{Q}) \Omega(\mathrm{Q}).$$

Similarly, we have $\mathcal{F}_v = \mathcal{F} \Lambda$. □

Corollary 4.7 (Poincaré Lemma)**.** *Let $\alpha := \omega\, du + \lambda\, dv$ be a differential one form on a simply connected domain $D \subset \mathbb{R}^2$. If $d\alpha = (\lambda_u - \omega_v) du \wedge dv = 0$, there exists a smooth function $f \colon D \to \mathbb{R}$ such that $df = \alpha$.*

Proof. Consider the equation $\varphi_u = \varphi \omega$, $\varphi_v = \varphi \lambda$ and apply Theorem 4.6 for $n = 1$. Letting $f = e^\varphi$, we have the desired function. □

Proof of Theorem 4.1. The uniqueness is already shown in Corollary 2.6. We show the existence. Consider the equation (3.1). with initial condition at $\mathrm{P} \in D$

$$\mathcal{F}(\mathrm{P}) := \begin{pmatrix} \sqrt{E_0} & F_0/\sqrt{E_0} & 0 \\ 0 & \sqrt{(E_0G_0 - F_0^2)/E_0} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $E_0 = E(\mathrm{P}), \dots$. Then by Theorem 4.6, there exists the unique solution $\mathcal{F}\colon D \to \mathrm{GL}(3,\mathbb{R})$. Write $\mathcal{F} = (\omega, \lambda, \nu)$. Then by the equation (3.1), $\omega_v = \lambda_u$, that is, $\mathbb{R}^3$-valued one form $\alpha = \omega\, du + \lambda\, dv$ is closed. Then by the Poincaré lemma (Corollary 4.7), there exists a smooth map $f\colon D \to \mathbb{R}^3$ such that $f_u = \omega$, $f_v = \lambda$. We show that f is the desired surface. Let

$$\mathcal{H} := {}^t\mathcal{F}\mathcal{F} = \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v & f_u \cdot \nu \\ f_v \cdot f_u & f_v \cdot f_v & f_v \cdot \nu \\ \nu \cdot f_u & \nu \cdot f_v & \nu \cdot \nu \end{pmatrix}, \qquad \bigl(\mathcal{F} = (f_u, f_v, \nu)\bigr).$$

Take an arbitrary $\mathrm{Q} \in D$ and a path γ joining P and Q. Then $\hat{\mathcal{H}} = \mathcal{H} \circ \gamma$ satisfies the linear ordinary equation

$$\frac{d\hat{\mathcal{H}}}{dt} = {}^t\hat{\Omega}\hat{\mathcal{H}} + \hat{\mathcal{H}}\hat{\Omega} \tag{4.8}$$

where $\hat{\Omega}(t)$ is as in (4.5). On the other hand,

$$\hat{\mathcal{H}}_0 = \mathcal{H}_0 \circ \gamma, \qquad \mathcal{H}_0 = \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a solution of (4.8) with same initial condition as $\hat{\mathcal{H}}$ (cf. Problem 4-1). Thus $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0$ by the uniqueness part of Theorem 4.2. Since Q is arbitrary, we have

$$f_u \cdot f_u = E,\ f_u \cdot f_v = F,\ f_v \cdot f_v = G,\ f_u \cdot \nu = f_v \cdot \nu = 0,\ |\nu| = 1.$$

Hence the entries of first fundamental form of f is E, F, G and ν is the unit normal vector. Then by (3.1), we can show that the entries of the second fundamental form are L, M and N.

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Exercises

4-1[H] Let $\mathit{\Omega}$ and $\mathit{\Lambda}$ be as in (3.1). Prove that

$$\mathcal{H} := \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfies the equation

$$\frac{\partial \mathcal{H}}{\partial u} = {}^t\mathit{\Omega}\mathcal{H} + \mathcal{H}\mathit{\Omega}, \qquad \frac{\partial \mathcal{H}}{\partial v} = {}^t\mathit{\Lambda}\mathcal{H} + \mathcal{H}\mathit{\Lambda}.$$

5 The Asymptotic Chebyshev Nets

Asymptotic directions. Let $f\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$ be an immersion and fix $\mathrm{P} = (u_0, v_0) \in D$. Consider a curve $\gamma(t) = f\bigl(u(t), v(t)\bigr)$ with $\gamma(0) = f(\mathrm{P})$. We define the *normal curvature* of $\gamma(t)$ at P as

$$\kappa_n(\gamma, \mathrm{P}) := \left(\frac{\ddot\gamma(0)}{|\dot\gamma(0)|^2}\right) \cdot \nu(\mathrm{P}), \tag{5.1}$$

where ν is the unit normal vector field of f.

Under the situations above, we have

$$\kappa_n(\gamma, \mathrm{P}) := \frac{L\,\dot u^2 + 2M\,\dot u \dot v + N\,\dot v^2}{E\,\dot u^2 + 2F\,\dot u \dot v + G\,\dot v^2}, \tag{5.2}$$

where E, F, G, L, M, and N are the entry of the first and second fundamental forms, which are evaluated at P, and $(\dot u, \dot v) = \bigl(\dot u(0), \dot v(0)\bigr)$.

In fact, by the chain rule, we have

$$\begin{aligned}
\dot\gamma(0) &= \left.\frac{d}{dt}\right|_{t=0} f\bigl(u(t), v(t)\bigr) = f_u \dot u + f_v \dot v,\\
\ddot\gamma(0) &= f_u \ddot u + f_v \ddot v + f_{uu}\dot u^2 + 2f_{uv}\dot u\dot v + f_{vv}\dot v^2,
\end{aligned}$$

where $\dot u$, $\ddot u$ etc. are evaluated at $t = 0$, and f_u, f_{uu} etc. are evaluated at P. Since f_u and f_v are perpendicular to ν and $L = f_{uu} \cdot \nu$, etc, we have (5.2). By (5.2), the normal curvature at P depends only on the velocity vector $\boldsymbol{v} = \dot\gamma(0)$ of $\gamma(t)$ at P. Moreover, it depends only on the direction of $\boldsymbol{v}$. So we write

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$$\kappa_n(\boldsymbol{v}) := \kappa_n(\gamma, \mathrm{P}), \qquad \boldsymbol{v} = \dot\gamma(0). \tag{5.3}$$

Theorem 5.1 (Proposition 9.5 in [5-1])**.** *The maximum and minimum of the normal curvature at* P *are the principal curvatures.*

Proof. Since $\kappa_n(\boldsymbol{v})$ depends only the direction of $\boldsymbol{v}$, then it can be considered as a function defined on S^1. Then it has the maximum and minimum. By (5.2), the maximum and minimum of κ_n are the maximum and minimum of

$$h(\alpha, \beta) := L\alpha^2 + 2M\alpha\beta + N\beta^2 \qquad \text{under the condition}$$
$$g(\alpha, \beta) := E\alpha^2 + 2F\alpha\beta + G\beta^2 = 1.$$

Let λ be the Lagrange multiplier. Then if κ_n takes maximum or minimum at (α, β) $(\neq (0, 0))$, $(h - \lambda g)_\alpha = (h - \lambda g)_\beta = 0$:

$$(L - \lambda E)\alpha + (M - \lambda F)\beta = 0, \qquad (M - \lambda F)\alpha + (N - \lambda G)\beta = 0.$$

This system admit a solution $(\alpha, \beta) \neq (0, 0)$ if and only if

$$\det\begin{pmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{pmatrix} = 0 \tag{5.4}$$

and in this case, $\lambda = \kappa_n$ is the maximum or minimum of $\kappa_n(\boldsymbol{v})$. Since (5.4) holds if and only if

$$\det\left[\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0,$$

that is, λ is an eigenvalue of A as in (1.13). Hence the maximum and minimum of κ_n are the principal curvatures. □

Corollary 5.2. *If the Gaussian curvature K is negative at* P*, there exists two linearly independent directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the tangent space at* P *such that $\kappa_n(\boldsymbol{v}_j) = 0$.*

Proof. Since $K(\mathrm{P}) < 0$, the principal curvatures λ_1 and λ_2, the maximum and the minimum of $\kappa_n(\boldsymbol{v})$, have opposite signs. □

Definition 5.3. The directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as in Corollary 5.2 is called the *asymptotic directions*.

Fact 5.4 (Theorem 9.9, Figure 8.1 in [5-1])**.** *At a point* P *with $K(\mathrm{P}) < 0$, the intersection of the surface and the tangent plane of the surface at* P *consists of two curves intersecting at* P*, and the tangent directions of these curves are the asymptotic directions.*

Fact 5.5 (Theorem B-5.4 in [5-1])**.** *Let* P *be a point on the surface with $K(\mathrm{P}) < 0$. Then there exists a local parameter (u, v) on a neighborhood U of* P *such that the u-direction and v-direction are the asymptotic directions on each point U.*

Definition 5.6. The coordinate system as in Fact 5.5 is called the *asymptotic coordinate system*.

Proposition 5.7. *A parameter (u, v) of the surface is asymptotic coordinate system if and only if the second fundamental form is in the form*

$$II = 2M\,du\,dv,$$

that is, $L = N = 0$.

Proof. Let $\mathrm{P} = (u_0, v_0)$. Then the normal curvature of the u-direction (resp. the v-direction) is $(f_{uu}/|f_u|^2) \cdot \nu = L/E$ (resp. $(f_{vv}/|f_v|^2) \cdot \nu = N/G$). The coordinate system (u, v) is asymptotic if and only if these two normal curvatures vanish, that is, $L = N = 0$. □

Example 5.8. Consider a parabolic hyperboloid $z = \frac{1}{2}(x^2 - y^2)$. Since this surface is parametrized as $(x, y) \mapsto \bigl(x, y, \frac{1}{2}(x^2 - y^2)\bigr)$, the first and second fundamental forms are

$$ds^2 = (1+x^2)\,dx^2 - 2xy\,dx\,dy + (1+y^2)\,dy^2, \quad II = \frac{dx^2 - dy^2}{\sqrt{1 + x^2 + y^2}}.$$

Since $dx^2 - dy^2 = (dx + dy)(dx - dy) = d(x + y)d(x - y)$, the parameter change $u = x + y$, $v = x - y$ yields

$$II = \frac{du\,dv}{\sqrt{1 + \frac{1}{2}u^2 + \frac{1}{2}v^2}}.$$

Hence (u, v) is the asymptotic coordinate system. The surface is represented by

$$(u, v) \longmapsto \left(\frac{u+v}{2}, \frac{u-v}{2}, \frac{uv}{2}\right).$$

Asymptotic Chebyshev net.

Theorem 5.9. *Let $f\colon \Sigma \to \mathbb{R}^3$ be an immersion of 2-dimensional manifold Σ into the Euclidean 3-space, whose Gaussian curvature is -1. Then for each point* P*, there exists a local coordinate* *system* (u, v) *on a neighborhood of* P *such that the first and second fundamental forms are represented by*

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad I\!I = 2\sin\theta\, du\, dv. \tag{5.5}$$

Here, θ *is a smooth function in* (u, v) *satisfying*

$$\theta_{uv} = \sin\theta. \tag{5.6}$$

The coordinate system (u, v) in Theorem 5.9 is called the *asymptotic Chebyshev net* and (5.6) is called the *sine Gordon equation*. Here function θ in (5.5) is the angle between the two asymptotic directions.

Proof. Let (u, v) be an asymptotic coordinate system around P (cf. Fact 5.5). Then the first and second fundamental forms are in the form

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2, \qquad I\!I = 2M\, du\, dv.$$

Then by Problem 3-1, the Codazzi equations yield

$$E_v = 0, \qquad G_u = 0.$$

Hence E and G depends only on u and v, respectively:

$$E = E(u), \qquad G = G(v).$$

Since E and G are positive, there exists a function $\xi = \xi(u)$, $\eta = \eta(v)$ such that

$$\xi_u = \sqrt{E(u)}, \qquad \eta_v = \sqrt{G(v)}.$$

Then (ξ, η) is the desired coordinate system. Then the fundamental forms are

$$ds^2 = d\xi^2 + 2\,\widetilde{F}\, d\xi\, d\eta + d\eta^2, \qquad I\!I = 2\widetilde{M}\, d\xi\, d\eta.$$

Since the Gaussian curvature is -1, that is,

$$K = \frac{-\widetilde{M}^2}{1 - \widetilde{F}^2} = -1,$$

we have

$$\widetilde{M}^2 + \widetilde{F}^2 = 1.$$

So there exists a smooth function θ such that $\widetilde{M} = \sin\theta$ and $\widetilde{F} = \cos\theta$. Thus we have the desired coordinate system. Moreover, by Problem 2-1, θ satisfies $\theta_{\xi\eta} = \sin\theta$ (which is equivalent to the Gauss equation). □

Remark 5.10. The asymptotic Chebyshev net is unique up to the coordinate changes

$$(u, v) \mapsto (\pm u + a, \pm v + b), \qquad (u, v) \mapsto (v, u).$$

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Exercises

5-1 Consider a smooth map $f\colon D \to \mathbb{R}^3$ as (cf. Problem 1-1)

$$f(u,v) = \left(\frac{\cos u}{\cosh v}, \frac{\sin u}{\cosh v}, v - \tanh v\right),$$

where $D = \{(u,v) \mid v > 0\}$.

(1) Write down the first fundamental and second fundamental forms in terms of (u,v).

(2) Find parameter change $(u,v) \mapsto (\xi,\eta)$ to the asymptotic Chebyshev net (ξ,η).

(3) Find the asymptotic angle $\theta(\xi,\eta)$.

6 Surfaces of constant negative curvature—the sine Gordon equation

Surfaces of constant negative curvature. As a corollary to Theorem 5.9 (the existence of asymptotic Chebyshev net) and the fundamental theorem for surface theory (Theorem 4.1), we have

Theorem 6.1. *For a function $\theta = \theta(u,v)$ defined on a simply connected region D on $\mathbb{R}^2$ satisfying $\theta_{uv} = \sin\theta$ and*

$$\theta(u,v) \in (0,\pi) \qquad \bigl((u,v) \in D\bigr) \tag{6.1}$$

there exists a unique immersion $f\colon D \to \mathbb{R}^3$ (up to congruence of $\mathbb{R}^3$) with first and second fundamental forms as

$$ds^2 = du^2 + 2\cos\theta\, du\, dv + dv^2, \qquad II = 2\sin\theta\, du\, dv. \tag{6.2}$$

Conversely, any surfaces in $\mathbb{R}^3$ with constant curvature -1 is obtained in this way.

As mentioned in Section 5, the equation

$$\theta_{uv} = \sin\theta. \tag{6.3}$$

Theorem 6.1 claims that the solutions of the sine-Gordon equation with

Example 6.2. Let

$$\theta(u,v) = 4\tan^{-1}\exp(u+v). \tag{6.4}$$

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Then one can easily see that it satisfies the sine-Gordon equation, and satisfies (6.1) on a domain $D = \{(u,v)\,|\,u+v<0\}$.

If we set $\xi := u-v$, $\eta := u+v$, the first and second fundamental forms can be written as

$$ds^2 = \frac{1}{\cosh^2\xi}(d\xi^2 + \sinh^2\eta\, d\eta^2), \quad II = \frac{\tanh\eta}{\cosh\eta}(-d\xi^2 + d\eta^2),$$

which coincide with the fundamental forms of the pseudosphere (Problem 1-1):

$$f(\xi,\eta) = \left(\frac{\cos\xi}{\cosh\eta}, \frac{\sin\xi}{\cosh\eta}, \eta - \tanh\eta\right).$$

The third fundamental form and the flat structure. Let $f\colon D \to \mathbb{R}^3$ be an immersion and $\nu\colon D \to S^2 \subset \mathbb{R}^3$ its unit normal vector field, where S^2 is considered as the set of unit vectors of $\mathbb{R}^3$.

Definition 6.3. The *third fundamental* form of f is the metric on D induced by the map ν:

$$III := d\nu\cdot d\nu := (\nu_u\cdot\nu_u)\,du^2 + 2(\nu_u\cdot\nu_v)\,du\,dv + (\nu_v\cdot\nu_v)\,dv^2,$$

where (u,v) is a local coordinate system on D.

Lemma 6.4. *The third fundamental form satisfies*

$$III - 2H\,II + K\,ds^2 = 0,$$

where H and K are the mean and the Gauss curvatures of f, and ds^2 and II are the first fundamental forms, respectively.

Proof. Fix a local coordinate system (u,v) and let $\widehat{I}$ and $\widehat{II}$ be the first and second fundamental matrices, respectively. Then the Weingarten matrix A is defined as $A := \widehat{I}^{-1}\widehat{II}$. Here, by the Weingarten formula (Theorem 2.1), it holds that

$$(\nu_u, \nu_v) = -(f_u, f_v)A.$$

Then the matrix representation (the third fundamental matrix) of $\widehat{III}$ is computed as

$$\begin{aligned}\widehat{III} &= \begin{pmatrix} {}^t\nu_u \\ {}^t\nu_v \end{pmatrix}(\nu_u, \nu_v) = {}^tA\begin{pmatrix} {}^tf_u \\ {}^tf_v \end{pmatrix}(f_u, f_v)A \\ &= {}^t\widehat{II}\,{}^t\widehat{I}^{-1}\,\widehat{I}\,\widehat{I}^{-1}\,\widehat{II} = \widehat{II}\,\widehat{I}^{-1}\,\widehat{II} = \widehat{I}\left(\widehat{I}^{-1}\,\widehat{II}\right)^2 = \widehat{I}A^2.\end{aligned}$$

On the other hand, by the Cayley-Hamilton formula we have

$$A^2 - (\operatorname{tr} A)A + (\det A)I = A^2 - 2HA + KI = O,$$

where I and O are the 2×2 identity matrix and the zero matrix, respectively. Thus, we have

$$O = \widehat{I}A^2 - 2H\,\widehat{I}A + K\,\widehat{I}\;\widehat{III} - 2H\,\widehat{II} + K\,\widehat{I},$$

and hence we have the conclusion. $\square$

Theorem 6.5. *Let $f\colon D \to \mathbb{R}^3$ be an immersion with constant Gaussian curvature -1, and let ν be its unit normal vector field. Then $ds^2 + III$ is a flat metric, that is, a Riemann metric of constant Gaussian curvature 0.*

Proof. Take the asymptotic Chebyshev net (u,v) as

$$ds^2 = du^2 + 2\cos\theta\,du\,dv + dv^2, \quad II = 2\sin\theta\,du\,dv.$$

Then the Weingarten matrix is expressed as

$$A = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 & \sin\theta \\ \sin\theta & 0 \end{pmatrix} = \begin{pmatrix} -\cot t & \csc t \\ \csc t & -\cot t \end{pmatrix},$$

and thus the mean curvature H is $-\cot t$. Thus, by Lemma 6.4,

$$\widehat{III} = -2\cot t\,\widehat{II} + \widehat{I} = \begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}.$$

Hence

$$\widehat{I} + \widehat{III} = 2I,$$

that is, $ds^2 + III = 2(du^2 + dv^2)$ which is a flat metric. $\square$

Remark 6.6. It is known that a complete, simply connected flat (with zero Gaussian curvature) Riemannian manifold (M, ds^2) is isometric to $\mathbb{R}^2$ with the canonical metric. We consider a complete immersion $f\colon M \to \mathbb{R}^3$ with constant Gaussian curvature. Since the induced metric ds^2 is complete, so is $d\sigma^2 := ds^2 + III$. Then the universal cover $(\widetilde{M}, d\tilde{\sigma}^2)$ of $(M, d\sigma^2)$ is isometric to the Euclidean plane.

Equations for the orthonormal frame. Let $f\colon D \to \mathbb{R}^3$ be a surface of constant Gaussian curvature -1 with unit normal

vector field ν, and (u,v) the asymptotic Chebyshev net with (6.2), We set
(6.5)
$$\boldsymbol{e}_1 := \frac{1}{2}\sec\frac{\theta}{2}(f_u+f_v), \quad \boldsymbol{e}_2 := \frac{1}{2}\csc\frac{\theta}{2}(-f_u+f_v), \quad \boldsymbol{e}_3 := \nu.$$

Then one can easily see that

$$\mathcal{G} := (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \tag{6.6}$$

is an orthogonal matrix for each (u,v). We call $\mathcal{G}$ the *orthonormal frame associated to the Chebyshev net* (u,v).

Lemma 6.7. *The orthonormal frame* (6.6) *satisfies*

$$\frac{\partial\mathcal{G}}{\partial u} = \mathcal{G}U, \quad \frac{\partial\mathcal{G}}{\partial v} = \mathcal{G}V, \tag{6.7}$$
$$U = \frac{1}{2}\begin{pmatrix} 0 & \theta_u & \sin\frac{\theta}{2} \\ -\theta_u & 0 & \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & -\cos\frac{\theta}{2} & 0 \end{pmatrix},$$
$$V = \frac{1}{2}\begin{pmatrix} 0 & -\theta_u & \sin\frac{\theta}{2} \\ \theta_v & 0 & -\cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \end{pmatrix}.$$

Proof. Direct computations from (6.5) and Theorem 2.5. Moreover, the integrability condition $U_v - V_u = UV - VU$ (cf. (4.4)) is equivalent to the sine-Gordon equation $\theta_{uv} = \sin\theta$. □

Extension of constant negative curvature surfaces. The advantage of (6.7) is that it is valid even if $\theta \equiv 0 \pmod{\pi}$. Thus, we have

Theorem 6.8. *Let $\theta\colon D \to \mathbb{R}^3$ be a smooth function on an simply connected domain D in the uv-plane satisfying the sine-Gordon equation* (6.3). *Then their exists a smooth map $f\colon D \to \mathbb{R}^3$ and $\nu\colon D \to S^2 \subset \mathbb{R}^3$ such that*

$$f_u\cdot\nu = 0, \quad f_v\cdot\nu = 0, \quad (\nu\cdot\nu = 1), \tag{6.8}$$

and

$$\begin{aligned} ds^2 &:= df\cdot df = du^2 + 2\cos\theta\,du\,dv + dv^2, \\ II &:= -d\nu\cdot df = 2\sin\theta\,du\,dv. \end{aligned} \tag{6.9}$$

Moreover, f is an immersion of constant Gaussian curvature -1 on the regions $\{(u,v) \mid \theta(u,v) \not\equiv 0 \pmod{\pi}\}$.

Proof. Since sine-Gordon equation is the integrability condition for (6.7). So there exists a solution $\mathcal{G}$ with the initial condition $\mathcal{G}(\mathrm{P}_0) = I$, where I is the identity matrix. Since both U and V are skew symmetric matrices, $\mathcal{G}$ takes its values the set of orthogonal matrices. In fact, one can easily show

$$(\mathcal{G}^t\mathcal{G})_u = (\mathcal{G}^t\mathcal{G})_v = O.$$

Let $\mathcal{G} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Then by the equation (6.7), the $\mathbb{R}^3$-valued 1-form

$$\omega := \left(\cos\frac{\theta}{2}\boldsymbol{e}_1 - \sin\frac{\theta}{2}\boldsymbol{e}_2\right)du + \left(\cos\frac{\theta}{2}\boldsymbol{e}_1 + \sin\frac{\theta}{2}\boldsymbol{e}_2\right)dv$$

is closed, that is, $d\omega = 0$. Then by the Poincaré Lemma (Corollary 4.7), there exists $f\colon D \to \mathbb{R}^3$ with $df = \omega$. This f is the desired one. □

Remark 6.9. Though the map $f\colon D \to \mathbb{R}^3$ has singular points on the set $\Sigma := \{(u,v) \in D \,|\, \theta(u,v) \equiv 0 \pmod{\pi}\}$, the unit normal vector field $\nu = \boldsymbol{e}_3$ is defined on Σ. A map $f\colon D \to \mathbb{R}^3$ is said to be a *frontal* if there exists a unit normal vector field $\nu\colon D \to S^2$, that is, ν satisfies (6.8). Moreover, if a smooth map $(f,\nu)\colon D \to \mathbb{R}^3 \times S^2$ is an immersion, f is called a *front* of a *wave front*. Various differential geometric properties for wave fronts are treated in [6-3], and will be treated in [6-2].

In these terms, our f in Theorem 6.8 is a front, because $ds^2 + III = 2(du^2 + dv^2)$ is positive definite, that is, (f,ν) is an immersion.

Example 6.10. The constant function $\theta(u,v) = 0$ satisfies the sine-Gordon equation (6.3). Then

$$\mathcal{G} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(u-v) & -\sin(u-v) \\ 0 & \sin(u-v) & \cos(u-v) \end{pmatrix}$$

is the solution of (6.7) with $\mathcal{G}(0,0) = I$. The corresponding map f is obtained as $f(u,v) = (u+v,0,0)$, that is, the image of f is the x-axis in $\mathbb{R}^3$. All points on the uv-plane are singular points.

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Exercises

6-1$^{\mathrm{H}}$ Consider the equation

$$(*) \qquad (\varphi - \theta)_u = 2a \sin\frac{\varphi+\theta}{2}, \qquad (\varphi+\theta)_v = \frac{2}{a}\sin\frac{\varphi-\theta}{2}$$

for an unknown φ, where $\theta = \theta(u,v)$ is a given function.

(1) Prove that, if θ satisfies the sine-Gordon equation (6.3), φ satisfies the sine Gordon equation, too.

(2) Find the general solution φ of (*) for $\theta = 0$.

7 Surfaces of constant negative curvature—Bäcklund Transformations

Bäcklund transformations. Let $f\colon D \to \mathbb{R}^3$ be a surface with unit normal vector field $\nu\colon D \to \mathbb{R}^3$. A surface $\hat{f}\colon D \to \mathbb{R}^3$ is said to be a *Bäcklund transformation* of f if there exists a positive constant r and an angle δ such that

(B-1) $\hat{f}(p) - f(p)$ is a tangent vector of both the surface f and $\hat{f}$ at $p \in D$,

(B-2) $|f(p) - \hat{f}(p)| = r$,

(B-3) the angle between $\nu(p)$ and $\hat{\nu}(p)$ is δ,

for each $p \in D$.

The following proposition gives a necesary condition for existence of Bäcklund transformations:

Proposition 7.1. *Assume hat there exists a Bäcklund transformation $\hat{f}$ of an immersion $f\colon D \to \mathbb{R}^3$. Then r and δ in (B-1) and (B-3) satisfy*

$$K = -\frac{\sin^2\delta}{r^2},$$

where K is the Gaussian curvature, that is, K is negative constant. Moreover, the Gaussian curvature $\hat{K}$ of the Bäcklund transformation $\hat{f}$ is the same constant as f.

To have constant negative Gaussian curvature is also the sufficient condition for existence of Bäcklund transformations:

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Proposition 7.2. *Let $f\colon D \to \mathbb{R}^3$ be a smooth map with unit normal vector field ν, where D is a simply connected domain of the uv-plane, and*

$$ds^2 := df\cdot df = du^2 + 2\cos\theta\,du\,dv + dv^2,$$
$$II := -df\cdot d\nu = 2\sin\theta\,du\,dv,$$

where $\theta = \theta(u,v)$ is a smooth function satisfying the sine-Grodon equation

$$\theta_{uv} = \sin\theta.$$

We fix $p_0 = (u_0, v_0) \in D$ and $\delta \in (0,\pi)$. Then

(B-1) for any $\varphi_0 \in \mathbb{R}$, there exists a unique solution of the differential equation

$$(\varphi-\theta)_u = 2\cot\frac{\delta}{2}\sin\frac{\varphi+\theta}{2}, \quad (\varphi+\theta)_v = 2\tan\frac{\delta}{2}\sin\frac{\varphi-\theta}{2}$$

with initial condition $\varphi(u_0, v_0) = \varphi_0$,

(B-2) for φ in (B-1), let

$$\hat{f} := f + \sin\delta\left(\cos\frac{\varphi}{2}\boldsymbol{e}_1 + \sin\frac{\varphi}{2}\boldsymbol{e}_2\right)$$
$$\hat{\nu} := \cos\delta\nu - \sin\delta - \sin\delta\left(\left(-\sin\frac{\varphi}{2}\boldsymbol{e}_1 + \cos\frac{\varphi}{2}\boldsymbol{e}_2\right)\right),$$

where

$$\boldsymbol{e}_1 := \frac{1}{2}\sec\frac{\theta}{2}(f_u + f_v), \quad \boldsymbol{e}_2 := \frac{1}{2}\csc\frac{\theta}{2}(-f_u + f_v).$$

Then $\hat{\nu}$ is a unit normal vector field of $\hat{f}\colon D \to \mathbb{R}^3$, and the first and second fundamtal forms are

$$d\hat{f} \cdot d\hat{f} := du^2 + 2\cos\varphi\, du\, dv + dv^2,$$
$$\widehat{I\!I} := 2\sin\varphi\, du\, dv.$$

That is, $\hat{f}$ is a Bäcklund transformation of f, and the asymptotic Chebyshev net of $\hat{f}$ coincides with that of f.

Exercises

7-1$^{\mathrm{H}}$ Describe how Dini's pseudosphere

$$f_{a,b}(u,v) := \left(\frac{a\cos u}{\cosh v}, \frac{a\sin u}{\cosh v}, a(v - \tanh v) + bu\right)$$

obtained as a Bäcklund transformation of the line

$$f_0(u,v) = (0, 0, u + v)$$

with unit normal vector field

$$\nu_0(u,v) = (-\sin(u-v), \cos(u-v), 0)\,.$$