3.1 What is a Linear Programming Problem?

Ex.1 Manufacture of toys

	Prices	Worth	Costs	Finishing	Carpentry
Wooden soldiers	$ 27	$ 10	$ 14	2 hours	1 hour
Wooden trains	$ 21	$ 9	$ 10	1 hour	1 hour

Conditions: no more than 100 hours of finishing hours weekly

no more than 80 hours of carpentry hours weekly

at most 40 demand of soldiers weekly

unlimited demand of trains

Find to maximize weekly profit

Solution

Decision Variables

x_1: number of soldiers produced each week

x_2: number of trains produced each week

Objective Function

Fixed costs do not depend on the value x1 and x2

Weekly revenues = $27 x_1 + 21 x_2$

Weekly raw material costs = $10 x_1 + 9 x_2$

Weekly variable costs = $14 x_1 + 10 x_2$

Weekly profit = $(27-10-14) x_1 + (21-9-10) x_2 = 3 x_1 + 2 x_2$

<u>Max $z = 3 x_1 + 2 x_2$</u>

Objective function coefficient

Constraints

Total finishing hrs. per week $= 2x_1 + 1x_2$ $2x_1 + x_2 \leq 100$

Total carpentry hrs. per week $= 1x_1 + 1x_2$ $x_1 + x_2 \leq 80$

At most 40 demand of soldiers per week $x_1 \leq 40$

Technological coefficient, Right-hand side (rhs)

Sign Restriction

Assume nonnegative values for decision variable

Optimization model

Max $z = 3x_1 + 2x_2$

Subject to (s.t.) $2x_1 + x_2 \leq 100$ $x_1 \geq 0$

$x_1 + x_2 \leq 80$ $x_2 \geq 0$

$x_1 \leq 40$

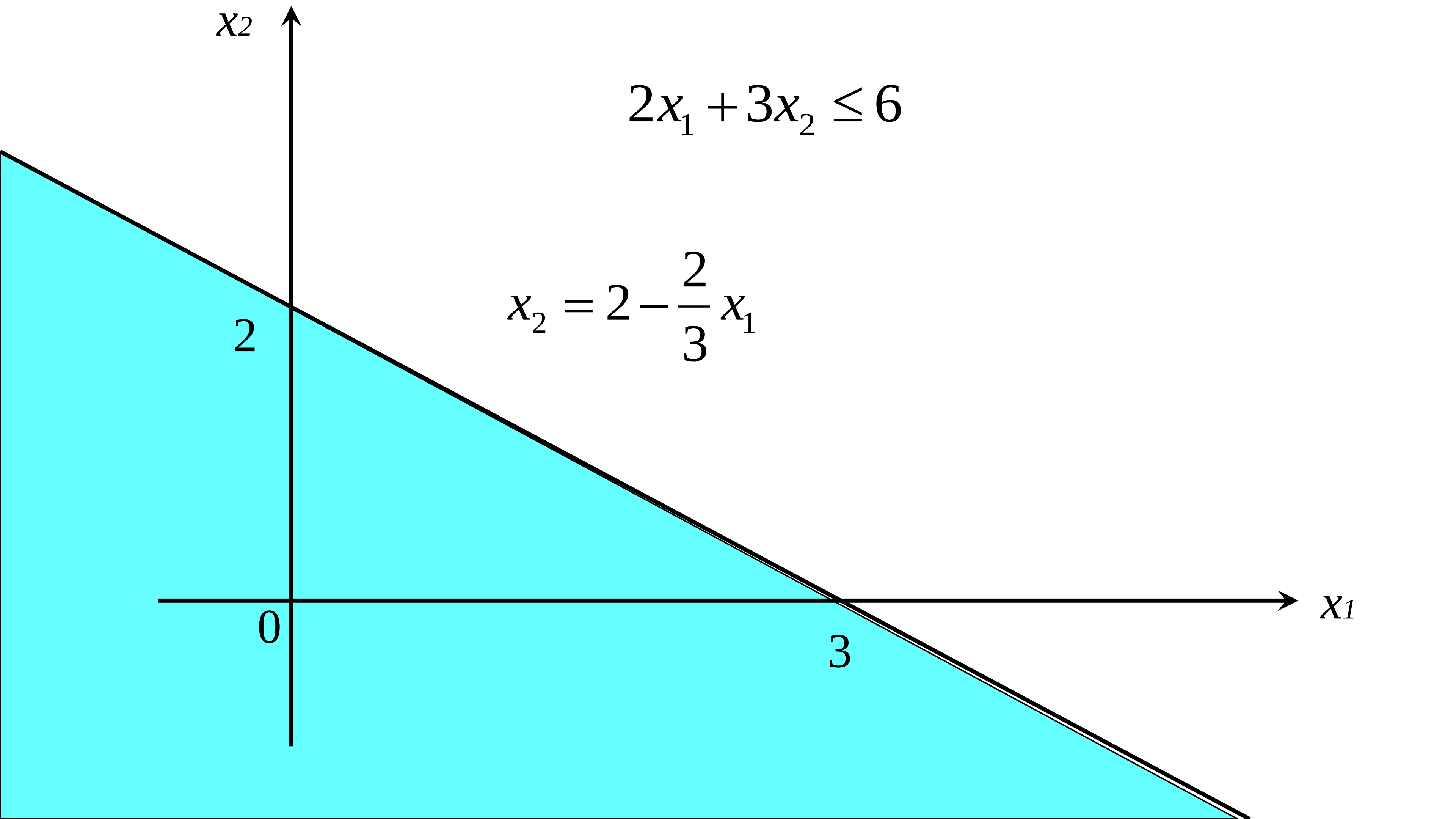
Assumption and Definition

1. Proportionality assumption of Linear Programming
2. Additivity assumption of Linear Programming
3. Divisibility assumption

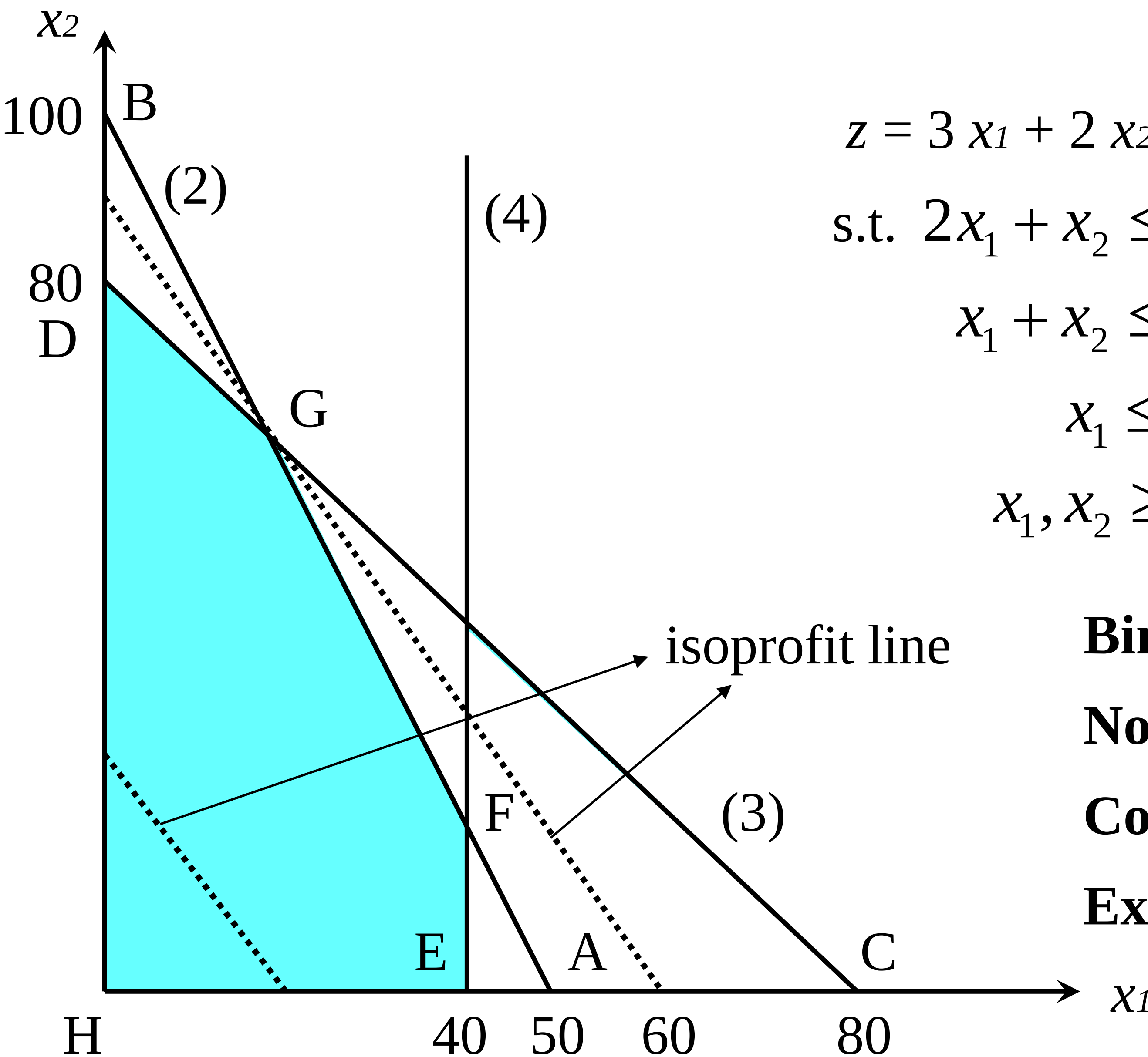
   --- Integer programming problem
4. Certainty assumption
5. Feasible region
6. Optimal solution

3.2 The Graphical Solution of Two-Variable

LP with only two variables can be solved graphically.



Finding the Feasible Solution



$z = 3\, x_1 + 2\, x_2$

$$\text{s.t.} \quad 2x_1 + x_2 \leq 100 \quad (2)$$

$$x_1 + x_2 \leq 80 \quad (3)$$

$$x_1 \leq 40 \quad (4)$$

$$x_1, x_2 \geq 0$$

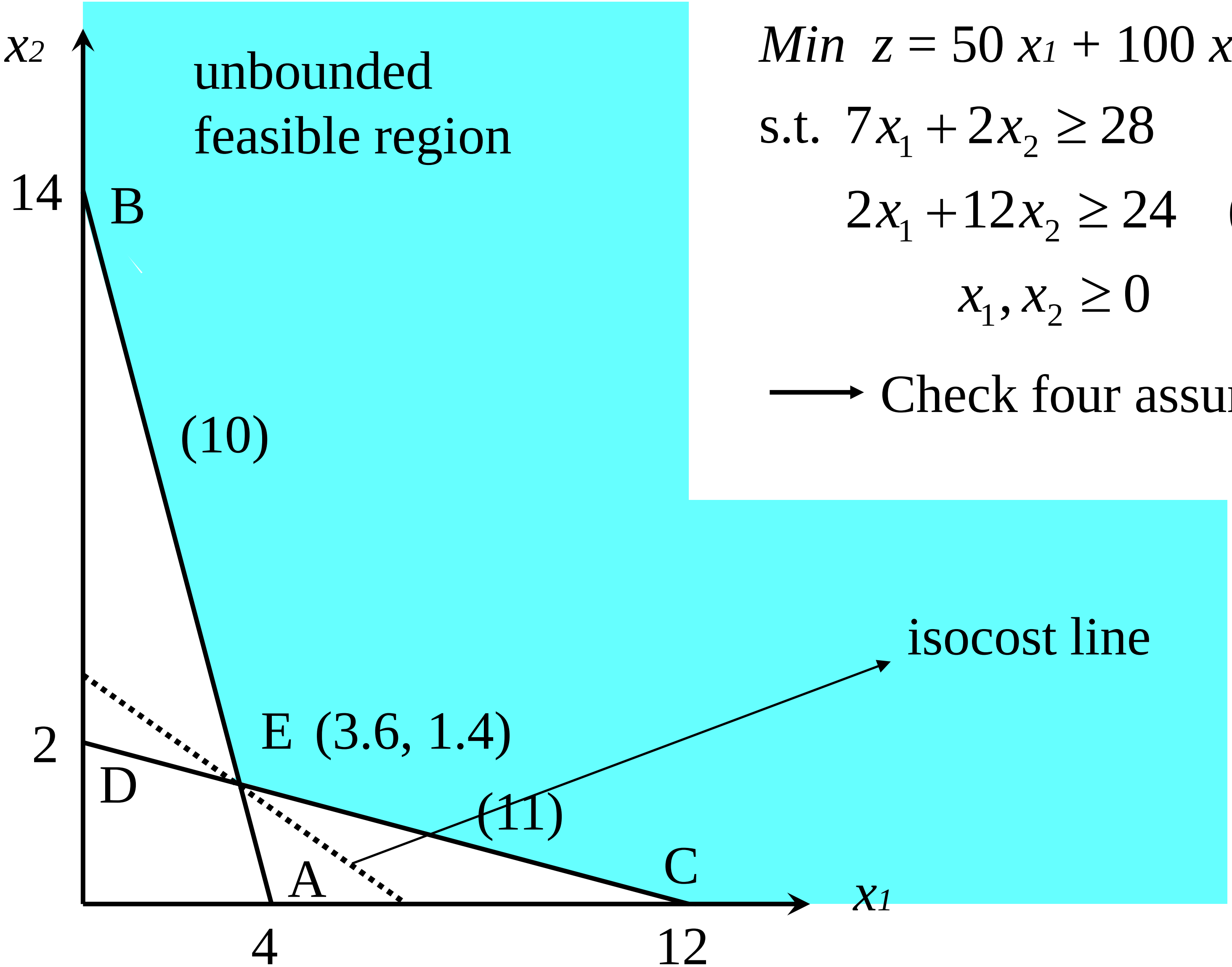
Binding

Nonbinding

Convex Set

Extreme point

Graphical Solution of Minimization Problems



$$Min \;\; z = 50\,x_1 + 100\,x_2$$

$$\text{s.t.} \;\; 7x_1 + 2x_2 \geq 28 \qquad (10)$$

$$2x_1 + 12x_2 \geq 24 \qquad (11)$$

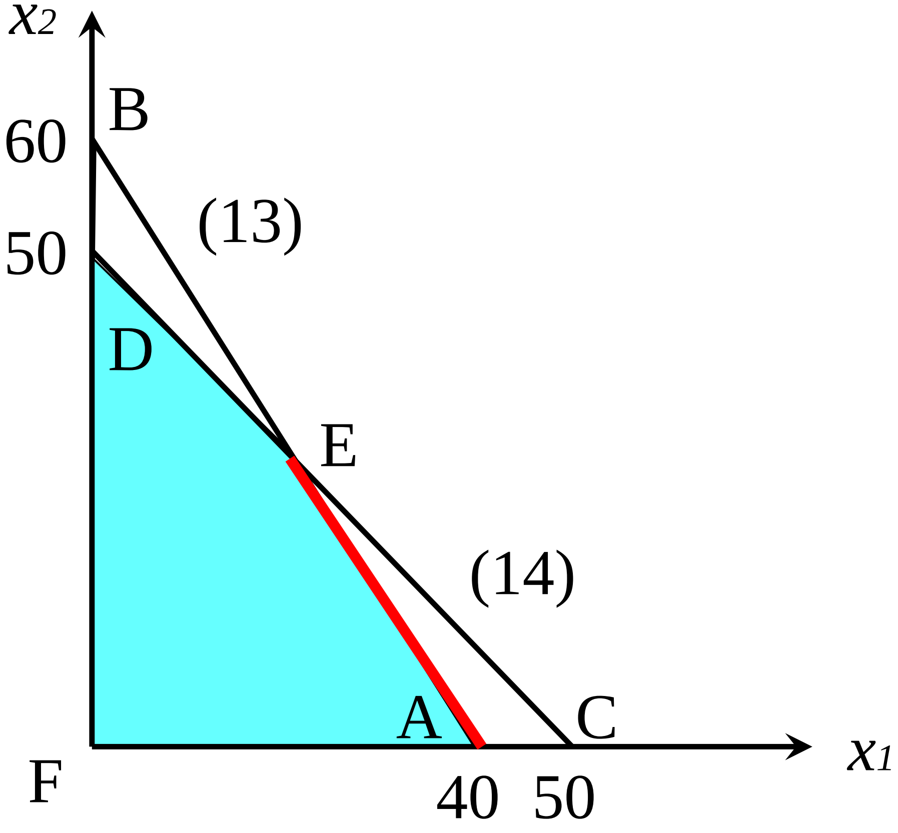
$$x_1, x_2 \geq 0$$

→ Check four assumption

3.3 Special Cases

Some types of LPs do not have unique optimal solution

An infinite number of optimal solutions
- Alternative or multiple optimal solutions



$$\max \; z = 3x_1 + 2x_2$$

$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1 \quad (13)$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1 \quad (14)$$

$$x_1, x_2 \ge 0$$

Infeasible

$max \;\; z = 3\,x_1 + 2\,x_2$

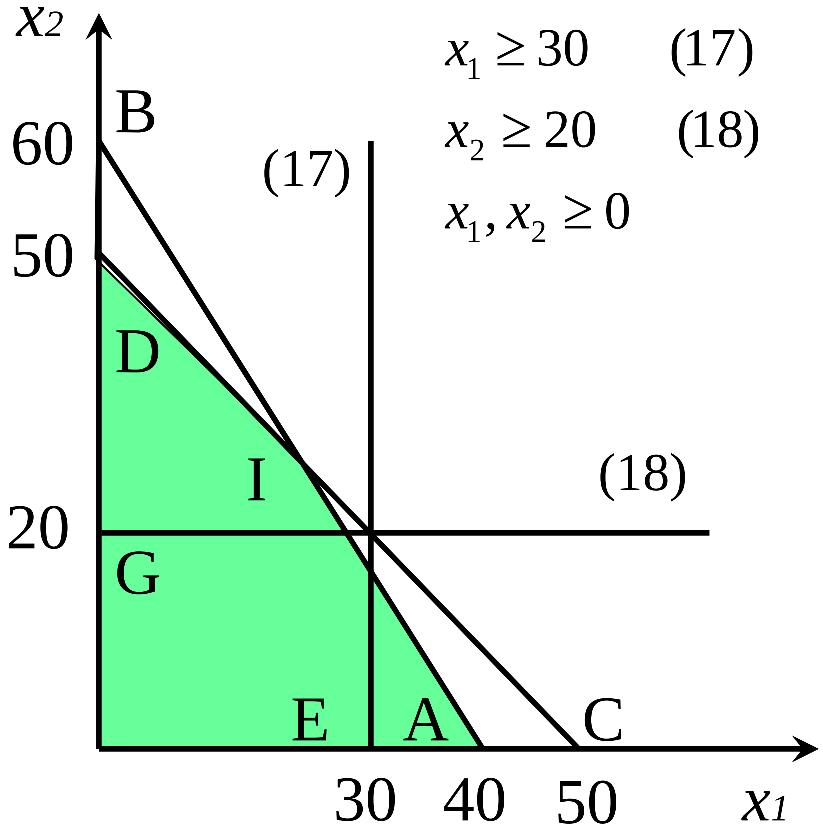
$$\text{s.t.} \quad \frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1$$

$$x_1 \ge 30 \qquad (17)$$

$$x_2 \ge 20 \qquad (18)$$

$$x_1, x_2 \ge 0$$



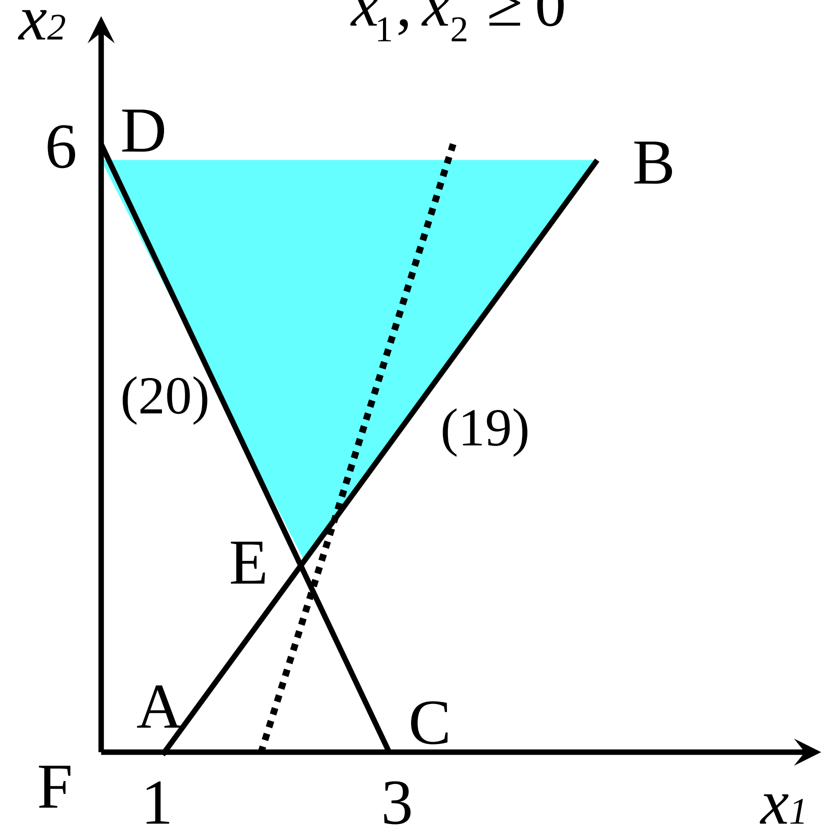
Unbounded

$max \;\; z = 2\,x_1 - x_2$

$$\text{s.t.} \quad x_1 - x_2 \le 1 \qquad (19)$$

$$2x_1 + x_2 \ge 6 \qquad (20)$$

$$x_1, x_2 \ge 0$$



3.4 Diet Problem

Satisfy daily nutritional requirement at minimum cots

$$\min \quad z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{aligned}
\text{s.t.} \quad & 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 && \text{Daily calorie intake at least 500} \\
& 3x_1 + 2x_2 \geq 6 && \text{Daily chocolate intake at least 6} \\
& 2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 && \text{Daily sugar intake at least 10} \\
& 2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 && \text{Daily fat intake at least 8} \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}$$

Optimal Solution

$$x_1, x_4 = 0, \quad x_2 = 3, x_3 = 1$$

$$z = 50x_1 + 20x_2 + 30x_3 + 80x_4 = 90$$

3.5 Work-Scheduling Problem

Post office to minimize the number of full-time employees

Incorrect solution

$$\min \quad z = x_1 + x_2 + \cdots + x_6 + x_7$$

x_i: number of employees working on day i

Day 1: Monday,
Day 2: Tuesday,…

s.t.

$$x_1 \geq 17$$
$$x_2 \geq 13$$
$$x_3 \geq 15$$
$$x_4 \geq 19$$
$$x_5 \geq 14$$
$$x_6 \geq 16$$
$$x_7 \geq 11$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

Correct solution

$$\min \quad z = x_1 + x_2 + \cdots + x_6 + x_7$$

x_i: number of employees beginning to work on day i

Day 1: Monday,
Day 2: Tuesday,…

s.t.

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 17$$
$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 13$$
$$x_1 + x_2 + x_3 + x_6 + x_7 \geq 15$$
$$x_1 + x_2 + x_3 + x_4 + x_7 \geq 19$$
$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 14$$
$$x_2 + x_3 + x_4 + x_5 + x_6 \geq 16$$
$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 11$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

3.6 Capital Budgeting Problem

Determine what fraction of each investment to purchase

$$\max \quad z = 13x_1 + 16x_2 + 16x_3 + 14x_4 + 39x_5$$

To maximize the NPV earned from investment

x_i: fraction of investment *i* purchased

$$\text{s.t.} \quad 11x_1 + 53x_2 + 5x_3 + 5x_4 + 29x_5 \le 40 \qquad \text{Cash flow in time 0}$$

$$3x_1 + 6x_2 + 5x_3 + x_4 + 34x_5 \le 20 \qquad \text{Cash flow in time 1}$$

$$x_1, x_2, x_3, x_4, x_5 \le 1 \qquad \text{Fraction condition}$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

*Net Present Value (NPV)   r: annual interest rate

\$1 now = \$$(1+r)^k$ *k* years from now

1 dollar *k* years from now is equivalent to receiving \$$(1+r)^{-k}$ now