Nash Equilibrium and Mixed Strategies (April 23)

I. Review

- Iterated removal of strictly dominated strategies leads to a unique outcome for some games but not for all games.
- The same could not be said for iterated removal of weakly dominated strategies. Sometimes, the order in which weakly dominated strategies are deleted mattered.
- Iterated removal of strategies that are never best responses – rationalizable strategies.
- The concepts of strict domination, weak domination, and never best response never considered the rationality of the other players' strategies in the definition itself.
- Today: a concept that can be used in all games but now involves the interaction among players

II. Definition of Nash Equilibrium

> A strategy combination $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is a **Nash equilibrium** if for each $i \in N$ and $s_i \in S_i$
> $$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

- Interpretations:[1]
  - One interpretation: Once $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is reached, no player $i \in N$ can obtain a higher payoff by changing his/her strategy to s_i. $\rightarrow$ self-enforcing
  - Another interpretation: A Nash equilibrium is an outcome reached through rational reasoning by the players. The rationality required to make sense of this interpretation is not clear but should be stronger than what was assumed in the last lecture. (As is shown later in Fact 1, for each $i \in N$ and Nash equilibrium s^*, s_i^* cannot be strictly dominated.)
  - Necessary condition: If theory were to determine a unique outcome for a game that is to be reached, it should be a Nash equilibrium.

[1] The discussion below is partially taken from Mas-Colell, Whinston, and Green (1995).

- However, there is no argument as to how a Nash equilibrium is reached. Moreover, a paper by Hart and Mas-Colell (2003) argues that there is no intuitive adjustment process that leads to a Nash equilibrium in general. This does not rule out the possibility for the result to hold under certain classes of games – potential games, supermodular games.

- For each $i \in N$, define the **best response correspondence** β_i in the following way:

$$\beta_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \; \forall s'_i \in S_i\}$$

- When for each s_{-i}, $\beta_i(s_{-i})$ is a singleton, β_i is sometimes called the **best response function**.

> **An equivalent definition:** $s^* = (s^*_1, s^*_2, \cdots, s^*_n)$ is a **Nash equilibrium** if for each $i \in N$, $s^*_i \in \beta_i(s^*_{-i})$

III. Finding Nash Equilibria – Review

- Prisoner's dilemma:

$1 \setminus 2$	C	D
C	$-2, -2$	$-6, \underline{0}$
D	$\underline{0}, -6$	$\underline{-5}, \underline{-5}$

Note: The underline indicates the best response choices for each player. For example, the underlined "0" in the entry (D, C) indicates that for player 1, choosing D is a best response to player 2 choosing C. The **strategy combination** for which both payoffs are underlined constitutes a Nash equilibrium. In this example, (D, D) is the **only Nash equilibrium**.

- Matching coins: Each player chooses heads (H) or tails (T) of a two-sided coin simultaneously. If the sides match – either both players choose H or both player choose T – player 1 receives a payoff of -1, while player 2 receives a payoff of 1. If they do not match, player 1 receives a payoff of 1, while player 2 receives a payoff of -1.

$1 \setminus 2$	H	T
H	$-1, \underline{1}$	$\underline{1}, -1$
T	$\underline{1}, -1$	$-1, \underline{1}$

Note that this game has no Nash equilibrium.

- Cournot duopoly: (simplified version and change in notation – changes are in red)
  - $N = \{1, 2\}$ (2 players who are called in this setting as firm 1 and firm 2)
  - $S_1 = S_2 = [0, \infty)$: production level of each firm (strategy sets can be infinite and unbounded)

$$u_1(s_1, s_2) = p(s_1, s_2)s_1 - cs_1$$

$$u_2(s_1, s_2) = p(s_1, s_2)s_2 - cs_2$$

    where
    * $p(s_1, s_2)$= $\max\{0, a-(s_1+s_2)\}$ denotes the inverse demand function giving the price of the output when firm 1 produces the amount s_1 and firm 2 produces the amount s_2.
    * c: (common) cost per unit production for both firm 1 and firm 2. Assume that $a > c$.
  - Given s_2, firm 1's best response function is given by

$$\beta_1(s_2) = \begin{cases} \frac{a-s_2-c}{2} & \text{if } a - s_2 - c > 0 \\ 0 & \text{if } a - s_2 - c \leq 0 \end{cases}$$

  - Given s_1, firm 2's best response function is given by

$$\beta_2(s_1) = \begin{cases} \frac{a-s_1-c}{2} & \text{if } a - s_1 - c > 0 \\ 0 & \text{if } a - s_1 - c \leq 0 \end{cases}$$

  - (s_1^*, s_2^*) is a Nash equilibrium if and only if

$$\beta_1(s_2^*) = s_1^* \text{ and } \beta_2(s_1^*) = s_2^*$$

  - Solving the pair of equations yields

$$s_1^* = s_2^* = \frac{a-c}{3}$$

  - There is an adjustment process that leads to the Nash equilibrium, called the Cournot tatonnement process.

IV. Properties

- The fact below summarizes the relationship between Nash equilibrium and the iterated removal of strictly dominated strategies.

Fact 1.

1. Let s^* be a Nash equilibrium. Then, for each $i \in N$, s_i^* is rationalizable, where rationalizability is defined in terms of Bernheim (1984) and Pearce (1984).
2. Let s^* be a Nash equilibrium. Then, for any $i \in N$, s_i^* cannot be deleted in the iterated removal of strictly dominated strategies.
3. Suppose that S_i is finite for each $i \in N$. If for each $i \in N$, s_i^* is the only strategy that remains after the iterated removal of strictly dominated strategies, then $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is the unique Nash equilibrium of the game. That is, for dominance solvable games, the iterated removal of strictly dominated strategies yields the unique Nash equilibrium.

- The second part of the above result does not hold when "strictly dominated" is replaced by "weakly dominated."
- The second part does not hold for games that are not dominance solvable (for example, the chicken game).

V. Mixed Strategies and Existence Theorem
From this section, suppose that S_i be a finite set for each $i \in N$.

- Each element in S_i is called a **pure strategy** of player i.
- A **mixed strategy** of player i is a function $\sigma_i : S_i \to \mathcal{R}$ such that
  - $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$
  - $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$

  where $\sigma_i(s_i)$ indicates the probability that player i plays the strategy s_i.
- $\Delta(S_i)$: the set of mixed strategies of player $i \in N$. (to be explained in further detail)
- Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n) \in \prod_{i \in N} \Delta(S_i)$. Under the assumption that players choose their mixed strategies independently, the probability that player 1 plays strategy

s_1, player 2 plays s_2, $\cdots$, player n plays s_n is given by

$$\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n) = \prod_{i\in N}\sigma_i(s_i)$$

- The **expected payoff** when each player i chooses a mixed strategy σ_i is given by

$$\pi_i(\sigma_1, \sigma_2, \cdots, \sigma_n) = \sum_{(s_1, s_2, \cdots, s_n)\in S}\left(\prod_{i\in N}\sigma_i(s_i)\right)u_i(s_1, s_2, \cdots, s_n)$$

  Recall $S := \prod_{i\in N} S_i$.

- $(N, (\Delta(S_i))_{i\in N}, (\pi_i)_{i\in N})$ defines a strategic form game and is called the **mixed extension** of the game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$.

- Nash equilibrium and concepts of strict domination can be defined for the mixed extension of the game in the same way.

  - $\sigma^* = (\sigma_1^*, \sigma_2^*, \cdots, \sigma_n^*) \in \prod_{i\in N}\Delta(S_i)$ is a **Nash equilibrium in mixed strategies** if for all $i \in N$ and $\sigma_i \in \Delta(S_i)$,

$$\pi_i(\sigma_i^*, \sigma_{-i}^*) \geq \pi_i(\sigma_i, \sigma_{-i}^*)$$

  - A mixed strategy $\sigma_i \in \Delta(S_i)$ is **strictly dominated** (version 1) by another mixed strategy $\sigma_i' \in \Delta(S_i)$ if for all $\sigma_{-i} \in \prod_{j\neq i}\Delta(S_j)$,

$$\pi_i(\sigma_i, \sigma_{-i}) < \pi_i(\sigma_i', \sigma_{-i})$$

- For a mixed extension of a finite game in strategic form, there always exists a Nash equilibrium in mixed strategies.

Theorem. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form where for each $i \in N$, S_i is a finite set. Let $G' = (N, \Delta(S_i)_{i\in N}, (\pi_i)_{i\in N})$ be the mixed extension of G. Then, there exists a Nash equilibrium $\sigma^* \in \prod_{i\in N}\Delta(S_i)$ of G'.

VI. Key Properties of Mixed Extensions

- A mixed strategy σ_i is strictly dominated by another mixed strategy σ_i' if and only if for all $s_{-i} \in S_{-i}$,

$$\pi_i(\sigma_i, s_{-i}) < \pi_i(\sigma_i', s_{-i}).$$

- Suppose that $s_i \in S_i$ is strictly dominated in $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. Then, any $\sigma_i \in \Delta(S_i)$ with $\sigma_i(s_i) > 0$ is strictly dominated in the mixed extension of G.
- Let σ_i be a best response to $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. Then, every $s_i \in S_i$ such that $\sigma_i(s_i) > 0$ is a best response to σ_{-i}.
- Suppose $s_i \in S_i$ is not strictly dominated by any other pure strategy $s_i' \in S_i$. The strategy $s_i \in S_i$ may be strictly dominated by a mixed strategy $\sigma_i \in \Delta(S_i)$.
- Suppose that $s_i \in S_i$ is not a best response to any combination of pure strategies $s_{-i} \in S_{-i}$. The strategy s_i may still be a best response to some combination of mixed strategies $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$.

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