Iterated Removal of Dominated Strategies (April 16)

I. Review

- Game in strategic form: $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ where
  - N: set of players
  - S_i: set of strategies of player $i \in N$
  - u_i: payoff function of player $i \in N$
- Strict domination:

Definition. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic form game. A strategy of player i, $s_i \in S_i$ is said to be **strictly dominated by** $s_i' \in S_i$ if for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

A strategy s_i is said to be **strictly dominated** if it is strictly dominated by some $s_i' \in S_i$.

- By eliminating strictly dominated strategies in the prisoner's dillema, only one strategy, D, remained for both players.

II. Iterated Removal of Strictly Dominated Strategies – Example

$1 \setminus 2$	L	C	R
U	$3,3$	$2,1$	$0,0$
M	$2,2$	$2,1$	$0,0$
D	$0,1$	$0,1$	$0,0$

- Suppose that both players are rational in that they do not choose strictly dominated strategies. Moreover, suppose that each player knows that the other player is rational.
- In the above example, none of the strategies for player 1 (U, M, D) are strictly dominated. So, supposing that player 1 is rational, player 1 still may choose U, M, or D.
- Strategy R of player 2 is strictly dominated by L and C. L nor C is strictly dominated. Therefore, player 2, if rational, will not choose R.

$1 \setminus 2$	L	C	R
U	$3, 3$	$2, 1$	$0, 0$
M	$2, 2$	$2, 1$	$0, 0$
D	$0, 1$	$0, 1$	$0, 0$

- Suppose that player 1 knows that player 2 is rational. Then, player 1 knows that player 2 will not choose R. Now, because both players know that player 2 will not choose R, the game is reduced to the following:

$1 \setminus 2$	L	C
U	$3, 3$	$2, 1$
M	$2, 2$	$2, 1$
D	$0, 1$	$0, 1$

- In the reduced game, D is now strictly dominated by M. So, if player 1 knows that player 2 is rational, player 1 will not choose D. If player 2 also knows that player 1 knows that player 2 is rational, then both players know that player 1 will not choose D, and the game is reduced to the following:

$1 \setminus 2$	L	C
U	$3, 3$	$2, 1$
M	$2, 2$	$2, 1$

- In the game above, strategy C is strictly dominated by L. Therefore, if player 2 knows that player 1 knows that player 2 is rational, then player 2 will not choose C. If player 1 also knows that player 2 knows that player 1 knows that player 2 is rational, then in both players' minds, the game is reduced to the following with now C deleted:

$1 \setminus 2$	L
U	$3, 3$
M	$2, 2$

- Now, M is strictly dominated by U. Therefore, if player 1 knows that player 2 knows that player 1 knows that player 2 is rational, then player 1 will not choose M. If player 2 also knows that player 1 knows that player 2 knows that player 1 knows that player 2 is rational, then both players can deduce that the strategy combination (U, L) results:

$1 \setminus 2$	L
U	$3, 3$

- The process described above $\rightarrow$ iterative removal of strictly dominated strategies
- Because the knowledge of rationality assumed for this process is complex, it is convenient instead to assume the following.

Common knowledge of rationality: Assume any chain (including infinite ones) of "Player 1 knows that player 2 knows that $\cdots$ (infinitely long)."

III. Iterated Removal of Strictly Dominated Strategies – General Procedure

- Suppose throughout this section that the set of strategies S_i for each $i \in N$ is finite.

Version 1:

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are strictly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are strictly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are strictly dominated.

- If G is a game such that the above process stops and yields a unique strategy combination $(s_1^*, s_2^*, \cdots, s_n^*) \in \prod_{i \in N} S_i$, then the game G is said to be **dominance solvable**.
- The following proposition shows an important property of this process for finite games.

Proposition 1. *Suppose that for each $i \in N$, S_i is a finite set. Let s_i be strictly dominated by some s_i' in step k. Then, s_i is also strictly dominated by some strategy s_i'' that is available in step l with $l \geq k$.*

- This result implies that it does not matter whether **all** strictly dominated strategies or just **one** strictly dominated strategy is deleted in one step.
- It also does not matter whether **all** players delete their strictly dominated strategies in one step or just **one** player deletes his/her strictly dominated strategies in one step.
- From above, we can define two alternative versions, both leading to the same set of strategies in the end.

Version 2:

1. Step 1: Choose **one** player $i \in N$ who has a strictly dominated strategy. Delete **one** $s_i \in S_i$ that is strictly dominated. Let S_i^1 denote the set of strategies that remain, and for the remaining players $j \neq i$, let $S_j^1 = S_j$.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$, choose **one** $i \in N$ and delete **one** $s_i \in S_i^1$ that is strictly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain for player i and let $S_j^2 = S_j^1$ for all other players $j \neq i$.
3. Continue the process until no player has a strictly dominated strategy.

IV. Iterated Removal of Weakly Dominated Strategies

- Review of definition of weak domination:

A strategy of player i, s_i is said to be **weakly dominated** by another strategy s_i' if for all $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i})$$

and for some s_{-i}, the above inequality holds with a strict inequality $<$. That is, for some $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

- If s_i is weakly dominated by s_i', then choosing s_i is **never better** and in at least one case **worse** than choosing s_i'.

- By replacing "strictly" with "weakly" in each version, one can think of anaologues of the two versions for strict domination.

Version 1W:

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are weakly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are weakly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are weakly dominated.

Version 2W:

1. Step 1: Choose one $i \in N$, and do the following. Delete all $s_i \in S_i$ that are weakly dominated. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Choose another $i \in N$, do the following. Considering now the game with S_i^1 as the set of strategies for each $i \in N$, delete $s_i \in S_i^1$ that are weakly dominated by some $s_i' \in S_i^1$. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there are no strategies that are weakly dominated.

- The strategies that remain after version 1W and 2W may not be the same, even if each player has a finite number of strategies – that is, even if S_i is a finite set for all players.
- The order in which the players are chosen in version 2W also affects which strategies remain in the end.

V. Never Best Response and Rationalizability

- A closely related concept to strict domination is concept of a strategy being never a best response.

Definition. Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form. A strategy $s_i \in S_i$ for player i is said to be a **best response** to $s_{-i} \in S_{-i}$ if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \ \forall s_i' \in S_i.$$

A strategy $s_i \in S_i$ is **never a best response** if there does not exist $s_{-i} \in S_{-i}$ to which s_i is a best response.

- If a strategy s_i is strictly dominated, then it is never a best response.
- Consider now iterated removal of strategies that are never best responses.

Iterated Removal of Strategies that are Never Best Responses:

1. Step 1: For all $i \in N$, delete all such $s_i \in S_i$ that are never best responses. Let S_i^1 denote the set of strategies that remain.
2. Step 2: Consider now the game with S_i^1 as the set of strategies for each $i \in N$. Delete all such $s_i \in S_i^1$ that are never best responses. Let S_i^2 denote the set of strategies that remain.
3. Continue the process until there is not a strategy that is never a best response.

- The end result is a set of strategies that are said to be **rationalizable**. See Bernheim (1984) and Pearce (1984).

VI. Reference Notes – These two books below can be found in the Department of Social Engineering library.

- Vega-Redondo (2003) (Chapter 1, pp.13-17; Chapter 2, pp. 30-35)
- Mas-Colell, Whinston, and Green (1995) (Chapter 7 Sections 7.A-B and 7.D, Chapter 8 Section 8.A-C)

References

Bernheim, B. D. (1984). Rationalizable strategic behavior. *Econometrica 52*(4), 1007–1028.

Mas-Colell, A., M. Whinston, and J. Green (1995). *Microeconomic Theory*. New York: Oxford University Press.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica 52*(4), 1029–1050.

Vega-Redondo, F. (2003). *Economics and the Theory of Games*. Cambridge University Press.