Advanced Macroeconomics (Department of Social Engineering, Spring FY2015):

Endogenous Growth Models (1)

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Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures)
  2. The Ramsey-Cass-Koopmans Model (4 lectures)
  3. Endogenous Growth Models (1-2 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic Dynamic Programming (3 lectures)

Introduction

- We now incorporate technological change into the model. The production function in this case takes the form:

$$Y(t) = F(K(t), A(t)L(t)),$$

where

- $\dot{A}(t)/A(t) = g > 0$
- $\hat{y}(t) \equiv Y(t)/(A(t)L(t))$, $\hat{k}(t) \equiv K(t)/(A(t)L(t))$, and $\hat{c}(t) \equiv C(t)/(A(t)L(t))$.

$\Downarrow$

Variables in per capita terms (c, k, y) grows at the *exogenous* rate of g in the steady state. (exercise)

Behavior of Households

- Consider a representative household with infinite horizon.
- His/her utility maximization problemF

$$\max_{c(t),a(t)} \quad U = \int_0^\infty \exp(-(\rho - n)t)\frac{c(t)^{1-\theta} - 1}{1-\theta}dt, \tag{1}$$

subject to

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \tag{2}$$

and

$$\lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s) - n)ds\right)\right] \geq 0, \tag{3}$$

and the given value of $a(0)$.

- We assume $\rho > n$.

Behavior of Households

- Current-value Hamiltonian is

$$\hat{H}(t, c, a, \mu) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + \mu(t)\big[(r(t) - n)a(t) + w(t) - c(t)\big]$$

- Conditions for utility maximization:

$$c(t)^{-\theta} = \mu(t), \qquad \dot{\mu}(t) = (\rho - r(t))\mu(t), \qquad \lim_{t\to\infty} \mu(t)a(t)\exp(-\rho t) = 0,$$

which yields the following consumption growth

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t) - \rho), \tag{4}$$

and the TVC:

$$\lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s) - n)ds\right)\right] = 0. \tag{5}$$

Behavior of Firms

- The only departure from the standard Ramsey model is that all firms have the linear production function.
- Then, aggregate output is $Y = AK$, which yields

$$y(t) = f(k(t)) = Ak(t). \tag{6}$$

- (6) means
  1. Marginal product of capital is NOT diminishing: $f''(k) = 0$.
  2. Inada condition is violated: $f'(k) = A$ for all $k \geq 0$.
- Conditions for profit maximization yields $A = R(t) \forall t$. Therefore

$$A - \delta = r(t). \tag{7}$$

Note: $w(t) = 0$.

Equilibrium

- Asset market equilibrium: $a(t) = k(t)$ ($\because$ closed economy).

- Substituting this, and firms' optimality conditions: $r(t) = A - \delta$ and $w(t) = 0$ into (2), (4) and (5):

$$\dot{k}(t) = (A - \delta - n)k(t) - c(t), \tag{8}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(A - \delta - \rho), \tag{9}$$

$$\lim_{t \to \infty} k(t) \exp[-(A - \delta - n)t] = 0. \tag{10}$$

- From (9), $\dot{c}(t)/c(t) =$ constant.

$$c(t) = c(0) \exp\left[\frac{1}{\theta}(A - \delta - \rho)t\right]. \tag{11}$$

Equilibrium

- We assume $A > \rho + \delta$ such that $\dot{c}(t)/c(t) > 0$.
- We assume that $(1-\theta)(A-\delta) + \theta n - \rho < 0$, such that the lifetime utility function given in (1) in equilibrium is bounded (prove that).
- In sum,

$$A - \delta > \rho > (1-\theta)(A-\delta) + \theta n. \tag{12}$$

If $\theta \to 1$, the second '$>$' is automatically satisfied since $\rho > n$.

- Hereafter, we let ψ denote

$$\psi \equiv -\frac{(1-\theta)(A-\delta) + \theta n - \rho}{\theta} > 0.$$

Equilibirum

- Note that $c(0)$ is still to be determined.
- Substituting $c(t)$ in (11) into (8):

$$\dot{k}(t) = (A - \delta - n)k(t) - c(0)\exp\left[\frac{1}{\theta}(A - \delta - \rho)t\right], \tag{13}$$

which is a first-order, linear differential equation in k.

- The general solution of this equation is

$$k(t) = \kappa \times \exp[(A - \delta - n)t] + \frac{c(0)}{\psi}\exp\left[\frac{1}{\theta}(A - \delta - \rho)t\right], \tag{14}$$

where $\kappa = k(0) - c(0)/\psi$.

Equilibrium

- Substituting the general solution (14) into the TVC (10),

$$\lim_{t\to\infty} \{k(t)\exp[-(A-\delta-n)t\} = 0$$
$$\Leftrightarrow \lim_{t\to\infty} \left\{\kappa + \frac{c(0)}{\psi}\exp(-\psi t)\right\} = 0$$

Since $\psi > 0$, the second term converges to 0.
$\Downarrow$
Hence, the TVC requires κ to be zero. $\Rightarrow c(0)$ is determined as $\psi k(0)$.

- Going back to (14):

$$k(t) = \underbrace{\kappa \times \exp[(A-\delta-n)t]}_{=0} + \underbrace{\frac{c(0)}{\psi}}_{=k(0)} \exp\left[\frac{1}{\theta}(A-\delta-\rho)t\right]$$
$$= k(0)\exp\left[\frac{1}{\theta}(A-\delta-\rho)t\right] \quad \therefore \frac{\dot{k}(t)}{k(t)} = \frac{1}{\theta}(A-\delta-\rho)\forall t \geq 0.$$

Equilibrium

- Since $\dot{c}(t)/c(t) = \dot{k}(t)/k(t)$ for all $t \geq 0$,

$$c(t) = \psi k(t) \ \forall t \geq 0.$$

  This means that there is NOT any transitional dynamics in contrast to Ramsey model.

- Note that $\dot{y}(t)/y(t) = (1/\theta)(A - \delta - \rho)$ since $y = Ak$.

- Then, the economy is on the balanced growth path (BGP) and three variables grow at the same rate of $(1/\theta)(A - \delta - \rho)$ from the initial time.

Equilibrium

- Note that the long-run growth rate is not only *sustained*, but also *endogenous* in the sense of being affected by deep parameters, ρ, θ, δ, A and so on.

- How is the saving rate affected by these parameters?

$$s = \frac{\dot{K}(t) + \delta K(t)}{Y(t)} = \frac{\dot{k}(t)/k(t) + n + \delta}{A}.$$

Since $\dot{k}(t)/k(t) = \dot{c}(t)/c(t) = (A - \delta - \rho)/\theta$,

$$s = \frac{A - \rho + \theta n + (\theta - 1)n}{\theta A}. \tag{15}$$

Summary of AK Model

- AK model gives a theoretical framework of sustained long-run growth, which is affected by parameters specifying preferences, technologies, and policies.
- Since all markets are competitive, there is a representative household, and there are no externalities, the equilibrium is Pareto optimal.
- However, there are some shortcomings. A major one is the share of capital in national income, say, RK/Y is equal to one in this model. However, in reality, that is about $1/3$.

A Model with Physical and Human Capital

- Assume that production requires physical capital K and human capital H:

$$Y = F(K, H) = f(\tilde{k})H, \quad \tilde{k} = K/H$$

- Assume that one unit of output can be used for
  1. one unit of consumption,
  2. one unit of investment for physical capital,

  or

  3. one unit of investment for human capital.

- Conditions for firms' profit maximization:

$$R(t) = f'(\tilde{k}(t)), \quad w(t) = f(\tilde{k}(t)) - \tilde{k}(t)f'(\tilde{k}(t)). \tag{16}$$

A Model with Physical and Human Capital

- Each household faces a opportunity to own two types of capital.
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- The no-arbitrage condition () of households:

$$R(t) - \delta = w(t) - \delta_H, \quad \delta_h \geq 0. \tag{17}$$

- From (16) and (17),

$$f'(\tilde{k}(t)) - \delta = f(\tilde{k}(t)) - \tilde{k}(t) f'(\tilde{k}(t)) - \delta_h.$$

$\widetilde{k}(t)$ becomes constant for all $t \geq 0$.

Endogenous Growth with Externalities

- Romer (1986) has constructed a model of endogenous growth in which spillover effects play a central role.
- Consider an economy without any population growth: i.e, $n = 0$ (important).
- The household side is essentially same as the standard Ramsey model, except $n = 0$.
- The production side of the economy consists of a set $[0, 1]$ of firms. The production function of firm $i \in [0, 1]$ is

$$Y_i(t) = F(K_i(t), A(t)L_i(t)). \tag{18}$$

Note: technology $A(t)$ is NOT indexed by i, but common to all firms.
$\Rightarrow$ Technology is assumed to be non-rival.

Endogenous Growth with Externalities

- Romer's assumption:

$$A(t) = BK(t), \tag{19}$$

where $K(t)$ is aggregate capital stock,

$$K(t) = \int_0^1 K_i(t)di. \tag{20}$$

- Three striking features:
  1. $A(t)$ is taken as given for each firm, but endogenously determined for the economy as a whole (Marshallian externalities).
  2. Aggregate stock $K(t)$ is a proxy of a technology, or knowledge stock (learning-by-doing).
  3. From a social perspective, the production function is

$$Y = F(K, BKL) = F(1, BL)K$$

If we define $A = F(1, BL)$, the model is reduced to a class of AK model.

Endogenous Growth with Externalities

- Firms are competitive, and takes $A(t)$ as given,
- Firm i's profit maximization:

$$\max_{K_i,L_i} F(K_i, AL_i) - RK_i - wL_i$$

Conditions for profit maximization for firm i:

$$R = \frac{\partial F(K_i, AL_i)}{\partial K_i},$$
$$w = A\frac{\partial F(K_i, AL_i)}{\partial (AL_i)},$$

which implies $K_i(t) = K(t)$, $L_i(t) = L(t)$.

- Let $L > 0$ denote labor supply (population of households), which is constant owing to $n = 0$.

Endogenous Growth with Externalities

- Since $F(\cdot,\cdot)$ is homogenous of degree 1,

$$\frac{Y}{K} = F(1, BL)$$

Let $\tilde{f}(L) = F(1, BL)$

- Then, we have

$$\tilde{f}'(L) = B\frac{\partial F(1, BL)}{\partial (BL)} = w(t)/K(t) \Leftrightarrow w(t) = \tilde{f}'(L)K(t),$$

and

$$\begin{aligned} R(t) &= \frac{Y(t)}{K(t)} - w(t)\frac{L}{K(t)} \\ &= \tilde{f}(L) - L\tilde{f}'(L). \end{aligned}$$

Endogenos Growth with Externalities

- Then, the growth rate in market equilibrium, denoted by g^* is

$$g^* = \frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{1}{\theta}(\tilde{f}(L) - Lf'(L) - \delta - \rho). \tag{21}$$

We assume that $g^* > 0$ and $(1-\theta)g^* < \rho + n$.

- On the other hand, the marginal product of capital form social perspective is

$$\frac{\partial Y}{\partial K} = \frac{\partial \tilde{f}(L)K}{\partial K} = \tilde{f}(L). \tag{22}$$

Then, the growth rate in the socially optimal allocation, denoted by g^S, is given by

$$g^S = \frac{1}{\theta}(\tilde{f}(L) - \delta - \rho) > g^*.$$

The growth rate is too low in the market equilibrium.