Advanced Macroeconomics

(Department of Social Engineering, Spring FY2015)

Ramsey-Cass-Koopmans Model (Full)

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Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures)
  2. The Ramsey-Cass-Koopmans Model (4 lectures)
  3. Endogenous Growth Models (1 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic DP (2-3 lectures)

Plan of Lectures in the Part of RCK Model

- May, 13: Optimal growth problem in discrete time
- May, 20: Optimal growth problem in continuous-time
- May, 27: Characterization of competitive equilibrium path
- June, 3: The government's activity

Reading List

- The followings are the lecture slides which provide an overview of the topics covered in
  1. Ch. 2 of Blanchard and Fischer (1989)
  2. Ch. 2 of Barro and Sala-i-Martin (2004)
  3. Ch. 8 of Acemoglu (2009)

(∗) Their full citations are collected in the syllabus of this course.

Introduction

- Recall that in the Solow-Swan model, consumption C and savings S are proportional to current income Y:

$$S = sY,\ C = (1 - s)Y,$$

where $s \in (0, 1)$ is the saving rate which is assumed to be exogenous.

- Ramsey-Cass-Koopmans model (or simply, Ramsey model):
  - This model differs from the Solow-Swan model in the respect that it endogenizes the savings rate by explicitly modeling the consumer's infinite-horizon dynamic optimization.

- Although the original Ramsey-Cass-Koopmans model is the model in continuous time, I will initially develop this model in discrete time.

Ramsey-Cass-Koopmans Model in Discrete Time

Setup

- The population L_t grows at rate $n > 0$:

$$L_{t+1} = (1+n)L_t.$$

  - The population is assumed to be equal to the labor force.
    $\Leftrightarrow$ Labor is assumed to be inelastically supplied.
  - $L_0 = 1$ is assumed.

- There is a single final good, which is produced using capital K and labor L.

$$Y_t = F(K_t, L_t),$$

Assumption

F is homogenous of degree one.

Setup

- The output is either consumed or invested.

- Let C_t denote the aggregate consumption. Then, the resource constraint is given by
$$(Y_t =)F(K_t, L_t) = C_t + K_{t+1} - (1-\delta)K_t,$$
where $\delta \in (0,1)$ is the capital depreciation rate.

- In per capita (一人当たり) terms,
$$f(k_t) = c_t + (1+n)k_{t+1} - (1-\delta)k_t, \tag{1}$$
where $f(k) \equiv F(k, 1)$.

(∗) lowercase letters denote per capita values of variables, say $k \equiv K/L$.

Setup

Assumption

1. $f'(k) > 0$,
2. $f''(k) < 0$,
3. $f(0) = 0$,
4. $\lim_{k\to 0} f'(k) = \infty$,
5. $\lim_{k\to\infty} f'(k) = 0$.

Optimal Growth Problem

- Consider a social planner (or central planner) who wants to maximize the households' welfare.
- The planner's problem is given as follows:

$$\begin{aligned} \max \quad & U_0 = \sum_{t=0}^{\infty} (1+n)^t \beta^t u(c_t) \\ \text{s.t.} \quad & (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \\ & k_0 > 0 \text{ given.} \end{aligned}$$

Assumption

$\beta(1+n) < 1.$

Optimal Growth Problem

- Therefore, in this problem, we assume away any market interactions.
- Instead, we consider an economy that a social planner directly determines the time paths of c_t and k_t so as to maximize the households' utility.

Optimal Growth Problem

- Let us define the following function v :

$$v(k_t, k_{t+1}) = u(f(k_t) + (1-\delta)k_t - (1+n)k_{t+1}).$$

Note that

$$v_1(k, k') = u'(c)[f'(k) + (1-\delta)] > 0, \quad v_2(k, k') = u'(c)[-(1+n)] < 0,$$

where $v_1 = \partial v/\partial k$ and $v_2 = \partial v/\partial k'$.

- Then, the problem is rewritten as the following *reduced form:*

$$\begin{aligned} \max \quad & \sum_{t=0}^{\infty} (1+n)^t \beta^t v(k_t, k_{t+1}) \\ \text{s.t.} \quad & k_{t+1} \in \Gamma(k_t) = \left\{ k \;\middle|\; k \in \left[0, \frac{f(k_t) + (1-\delta)k_t}{1+n}\right] \right\} \\ & k_0 > 0 \text{ given} \end{aligned}$$

Optimal Growth Problem

- The Bellman equation:

$$V(k) = \max_{k' \in \Gamma(k)} \{v(k, k') + \beta(1+n)V(k')\}$$

- If $u(c) = \ln c$, $f(k) = Ak^\alpha$, $n = 0$, and $\delta = 1$, we can explicitly obtain the value function V.
  (problem 4 of the problem set $\#1$).

Optimal Growth Problem

- We assume that V is differentiable. Then, the F.O.C is given by

$$v_2(k, k') + \beta(1+n)V_k(k') = 0.$$

where $V_k = \partial V(k)/\partial k$.

- Since $v_2 = -(1+n)u'(c)$,

$$u'(c) = \beta V_k(k'). \tag{2}$$

This gives the policy function implicitly, denoted by $k' = h(k)$.

Euler Equation

- Substituting this policy function back into the Bellman equation,

$$V(k) = v(k, h(k)) + \beta V(h(k))$$

- Then, by the envelope theorem,

$$V_k(k) = v_1(k, h(k)) = u'(c)(f'(k) + 1 - \delta) \tag{3}$$

- Using (3), the F.O.C(2) is rewritten as

$$\begin{aligned} u'(c) &= \beta V_k(k') \\ &= \beta u'(c')(f'(k') + 1 - \delta), \end{aligned}$$

or, if we use the time subscript,

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta). \tag{4}$$

(4) is the Euler equation in the discrete-time Ramsey-Cass-Koopmans model.

Intuition behind the Euler Equation

- What is the economic intuition behind the Euler equation?
  $\Rightarrow$ If the consumption path is optimal, the planner must be indifferent between the two alternatives:

1. Consume a unit of the final good today (period t), which attains the marginal utility gain by $u'(c_t)$.

2. Invest the same unit of the good instead.
   $\Downarrow$
   1. $f'(k_{t+1}) + 1 - \delta$ units of the good are additionally available for consumption.
   2. Since population grows at a rate n, $\frac{f'(k_{t+1})+1-\delta}{1+n}$ units of the good become available for one household.
   3. This induces the marginal utility tommorow (period $t+1$) by $\beta(1+n)u'(c_{t+1}) \times \frac{f'(k_{t+1})+1-\delta}{1+n} = \beta(f'(k_{t+1}) + 1 - \delta)u'(c_{t+1})$

Optimal Growth Path

- The Euler equation is the necessary condition for optimality, but not sufficient for that by itself.
- In the Ramsey-Cass-Koopmans model, the optimal growth path is given by $\{(k_t, c_t)\}_{t=0}^{\infty}$ such that

$$k_{t+1} = (1+n)^{-1}[f(k_t) + (1-\delta)k_t - c_t] \quad (1)$$

$$u'(c_t) = \beta(f'(k_{t+1}) + 1 - \delta)u'(c_{t+1}), \quad (4)$$

$$\lim_{t\to\infty} \beta^t v_1(k_t, k_{t+1})k_t = \lim_{t\to\infty} \beta^t u'(c_t)(f'(k_t) + 1 - \delta)k_t = 0. \quad (5)$$

($*$) By using the Euler equation (4), the transversality condition (5) is more simply expressed as

$$\lim_{t\to\infty} \beta^t u'(c_t)k_{t+1} = 0.$$

Steady State

- Let $(\overline{k}, \overline{c})$ denote the steady state.
- From the Euler equation (4),

$$1 = \beta(f'(\overline{k}) + 1 - \delta) \Leftrightarrow f'(\overline{k}) = 1/\beta - (1 - \delta) > 0. \tag{6}$$

  Owing to the Inada condition, we can readily show that there uniquely exists $\overline{k} > 0$ such that the above equation is satisfied.

- Once we have obtained $\overline{k}$,

$$\begin{aligned} \overline{c} &= f(\overline{k}) + (1 - \delta)\overline{k} - (1 + n)\overline{k} \\ &= f(\overline{k}) - (n + \delta)\overline{k}. \end{aligned}$$

Local Stability of Linearized System

- Linearized system around the steady state $(\overline{k}, \overline{c})$:

$$\begin{pmatrix} k_{t+1} - \overline{k} \\ c_{t+1} - \overline{c} \end{pmatrix} = J \begin{pmatrix} k_t - \overline{k} \\ c_t - \overline{c} \end{pmatrix},$$

- In the above equation, the Jacobian matrix J is

$$\begin{pmatrix} [\beta(1+n)]^{-1} & -(1+n)^{-1} \\ \dfrac{f''(\overline{k})\overline{c}}{(1+n)\varepsilon_u(\overline{c})} & 1 - \beta \dfrac{f''(\overline{k})\overline{c}}{(1+n)\varepsilon_u(\overline{c})} \end{pmatrix},$$

where $\varepsilon_u(c) = -\dfrac{cu''(c)}{u'(c)} > 0$ is the degree of relative risk aversion (相対的危険回避度).

Local Stability of Linearized System

- Characteristic equation:

$$p(\xi) \equiv \xi^2 - (\mathrm{tr}J)\xi + \det J = 0. \tag{7}$$

- Note that

$$p(0) = \det J = [\beta(1+n)]^{-1} > 0,$$
$$p(1) = 1 - \mathrm{tr}J + \det J = \beta \frac{f''(\bar{k})\bar{c}}{(1+n)\varepsilon_u(\bar{c})} < 0.$$

Then,

- Both roots are positive
- One root is strictly less than one, while the other is strictly greater than one.

That is, $(\bar{k}, \bar{c})$ is a saddle.

Local Stability of Linearized System

- Then, the general solution is

$$\begin{pmatrix} k_t - \bar{k} \\ c_t - \bar{c} \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} (\xi_1)^t + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} (\xi_2)^t, \qquad (8)$$

where

1. ξ_j $(j = 1, 2)$ is the eigenvalue of matrix J.
   $\Rightarrow$ Without any loss of generality, let $0 < \xi_1 < 1$ and $\xi_2 > 1$.
2. $v_j \equiv (v_{1j}, v_{2j})^T$ is the eigenvector corresponding to the eigenvalue ξ_j.
3. z_j $(j \in \{1, 2\})$ is a constant value *still to be determined*.

Determination of c_0

- Imposing $t = 0$ in (8):

$$\begin{pmatrix} k_0 - \overline{k} \\ c_0 - \overline{c} \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}. \tag{9}$$

- Because the initial value of physical capital in per capita, k_0 is exogenously given, z_1 and z_2 must satisfy

$$z_1 = \frac{k_0 - \overline{k} - v_{12} z_2}{v_{11}}.$$

Determination of c_0

- Thus, the initial consumption, c_0, must be determined such that

$$z_2 = 0,$$

which leads $z_1 = (k_0 - \overline{k})/v_{11}$.

- Therefore, from Eq. (8), we can analytically obtain the optimal growth path as follows:

$$k_t - \overline{k} = (k(0) - \overline{k})(\xi_1)^t, \quad c_t - \overline{c} = (v_{21}/v_{11})(k(0) - \overline{k})(\xi_1)^t. \tag{10}$$

Ramsey-Cass-Koopmans Model in Continuous Time

Optimal Growth Problem in Continuous Time

- Consider an economy that a social planner directly determines the time paths of $c(t)$ so as to maximize

$$\begin{aligned} \max \quad & \int_0^\infty \exp(-(\rho - n)t)u(c(t))dt, \\ \text{s.t.} \quad & \dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t), \\ & k(0) > 0 \text{ given.} \end{aligned} \tag{11}$$

Assumption

$\rho > n$.

Optimal Growth Problem in Continuous Time

- Hereafter, "(t)" is omitted unless to do so would cause some confusions.
- Let μ denote the costate variable Set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c) + \mu[f(k) - (n + \delta)k - c].$$

- F.O.Cs and the TVC:

$$\hat{H}_c(k, c, \mu) = 0 \Leftrightarrow u'(c) = \mu, \tag{12}$$

$$\dot{\mu} = (\rho - n)\mu - \hat{H}_k(k, c, \mu) \Leftrightarrow \dot{\mu} = -(f'(k) - \delta - \rho)\mu, \tag{13}$$

$$\lim_{t\to\infty}[e^{-(\rho-n)t}\mu(t)k(t)] = 0. \tag{14}$$

Euler Equation and Transversality Condition

- From (12) and (13)

$$\frac{cu''(c)}{u'(c)}\frac{\dot{c}}{c} = \frac{\dot{\mu}}{\mu} \Leftrightarrow \varepsilon_u(c)\frac{\dot{c}}{c} = -\frac{\dot{\mu}}{\mu}, \tag{15}$$

where $\epsilon_u(c) = -\dfrac{cu''(c)}{u'(c)} > 0.$

- Then,

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho). \tag{16}$$

($*$) (16) is the Euler equation in the continuous-time Ramsey model. This is also called the Keynes-Ramsey rule (ケインズ・ラムゼイルール).

Euler Equation and Transversality Condition

- If the terminal date is finite, the TVC is

$$e^{-(\rho-n)T}\mu(T)k(T) = 0.$$

- Thus, the TVC given by (14) is the limit as T tends to infinity.
- The intuition of the TVC: *Either* of the following three cases must be true:
  1. $k(t)$ asymptotically becomes 0;
  2. $k(t)$ remains positive, and the value of $\mu(t)$ approaches 0;
  3. $k(t)$ grows forever at a positive rate, and the the value of $\mu(t)$ approaches 0 at a faster rate than $\dot{k}/k - \rho$.

Euler Equation and Transversality Condition

- From (13),

$$\mu(t) = \mu(0) \exp\left(- \int_0^t [f'(k(s)) - (\delta + \rho)] ds \right). \tag{17}$$

- Substituting (17) into (14), we can obtain another form of the TVC as follows:

$$\lim_{t \to \infty} k(t) \exp\left(- \int_0^t [f'(k(s)) - (n + \delta)] ds \right) = 0. \tag{18}$$

Optimal Growth Path

- Therefore, we can obtain the optimal growth path $(k(t), c(t))$ by solving

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t), \tag{11}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}[f'(k(t)) - \delta - \rho], \tag{16}$$

$$\lim_{t\to\infty} k(t)\exp\left(-\int_0^t [f'(k(s)) - (n+\delta)]ds\right) = 0. \tag{18}$$

Steady State

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}(t) = 0.$$

- From the Euler equation with $\dot{c} = 0$,

$$\frac{\dot{c}(t)}{c(t)} = 0 \Leftrightarrow \frac{1}{\varepsilon_u(c(t))}(f'(k(t)) - \delta - \rho) = 0. \tag{19}$$

- Then, $\overline{k}$ is determined as

$$f'(\overline{k}) = \rho + \delta > n + \delta. \tag{20}$$

- Eq. (20) pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate

Steady State

- Once, $\overline{k}$ is determined,

$$\overline{c} = C(\overline{k}) \equiv f(\overline{k}) - (n+\delta)\overline{k}. \tag{21}$$

- Note that

$$C'(k) = f'(k) - (n+\delta) \gtreqless 0 \Leftrightarrow f'(k) \gtreqless n+\delta.$$

- Let us define k_{gold} such that $f'(k_{gold}) = n + \delta$.
  k_{gold} is golden rule of capital stock.

- Since $f'' < 0$,

$$C'(k) \gtreqless 0 \Leftrightarrow k \lesseqgtr k_{gold}.$$

Modified Golden Rule

- $\overline{k}$ is called the modified golden rule, which is smaller than the golden rule.
- Note that $\overline{k}$
  - maximizes life-time utility, because it is derived by solving the utility maximization problem,
  - does NOT maximize steady-state consumption (to see why, see figure)

This implies that achieving the golden rule is not desirable from the viewpoint of utility maximizing.

Transitional Dynamics

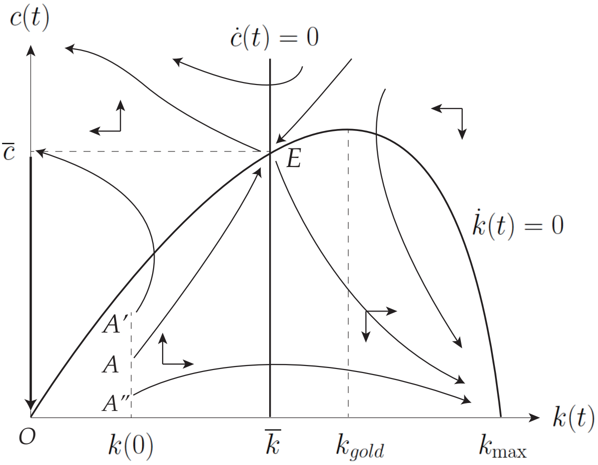
- From the dynamics of $k(t)$,

$$\dot{k}(t) \gtreqless 0 \Leftrightarrow c(t) \lesseqgtr f(k(t)) - (n+\delta)k(t) \equiv C(k(t)).$$

- From the dynamics of $c(t)$ with $c(t) > 0$,

$$\dot{c}(t) \gtreqless 0 \Leftrightarrow k(t) \lesseqgtr \overline{k}.$$

Transitional Dynamics



Local Stability of Linearized System

- The dynamics of $k(t)$: $\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t)$.
- Linearizing this equation in the neighborhood of the steady state $(\overline{k}, \overline{c})$, we have

$$\dot{k}(t) = [f'(\overline{k}) - (n+\delta)](k(t) - \overline{k}) - (c(t) - \overline{c}). \tag{22}$$

Since $f'(\overline{k}) = \rho + \delta$ in the steady state,

$$\dot{k}(t) = (\rho - n)(k(t) - \overline{k}) - (c(t) - \overline{c}). \tag{23}$$

Local Stability of Linearized System

- The dynamics of $c(t)$:

$$\dot{c}(t) = \frac{c(t)}{\varepsilon_u(c(t))}[f'(k(t)) - \delta - \rho].$$

- Linearizing this equation in the neighborhood of the steady state $(\overline{k}, \overline{c})$, we have

$$\dot{c}(t) = \frac{\overline{c} f''(\overline{k})}{\varepsilon_u(\overline{c})}(k(t) - \overline{k}). \tag{24}$$

Local Stability of Linearized System

- Therefore, linearized (or local) dynamics is given by

$$\dot{k}(t) = (\rho - n)(k(t) - \overline{k}) - (c(t) - \overline{c}), \tag{25}$$

$$\dot{c}(t) = \frac{\overline{c} f''(\overline{k})}{\varepsilon_u(\overline{c})}(k(t) - \overline{k}). \tag{26}$$

- Using a matrix form, the above system of equations is rewritten as

$$\begin{pmatrix} \dot{k}(t) \\ \dot{c}(t) \end{pmatrix} = J \begin{pmatrix} k(t) - \overline{k} \\ c(t) - \overline{c} \end{pmatrix}. \tag{27}$$

where J is Jacobian matrix:

$$J = \begin{pmatrix} \rho - n & -1 \\ \dfrac{f''(\overline{k})\overline{c}}{\varepsilon_u(\overline{c})} & 0 \end{pmatrix}. \tag{28}$$

Local Stability of Linearized System

- It is well known that the general solution is

$$\begin{pmatrix} k(t)-\overline{k} \\ c(t)-\overline{c} \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \exp(\xi_1 t) + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \exp(\xi_2 t), \tag{29}$$

where

1. ξ_j $(=1,2)$ is the eigenvalue of matrix J;
2. $v_j \equiv (v_{1j}, v_{2j})^T$ is the eigenvector corresponding to the eigenvalue ξ_j $(j \in \{1,2\})$;
3. z_j $(j \in \{1,2\})$ is a constant value *still to be determined*: i.e., z_j is endogenous.

Charateristic Equation

- ξ_j is determined from the following *characteristic equation:*

$$\det(J-\xi I)=0 \Leftrightarrow \det\begin{pmatrix} \rho-n-\xi & -1 \\ \dfrac{f''(\overline{k})\overline{c}}{\varepsilon_u(\overline{c})} & 0-\xi \end{pmatrix}=0,$$

$$\Leftrightarrow \xi^2-(\rho-n)\xi+\frac{f''(\overline{k})\overline{c}}{\varepsilon_u(\overline{c})}=0.$$

- It is shown that there are two real eigenvalues, one negative and one positive (consider the reason why).
- Without any loss of generality, let $\xi_1<0$ and $\xi_2>0$ respectively denote the positive and the negative eigenvalues.

Determination of Initial Consumption

- Determination of $c(0)$:

$$\begin{pmatrix} k(0)-\overline{k} \\ c(0)-\overline{c} \end{pmatrix} = z_1 \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} + z_2 \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} \tag{30}$$

- Thus, the initial consumption, $c(0)$, must be determined such that
  1. $z_2 = 0$: otherwise the economy diverges from the steady state, and such a path violates either the Keynes-Ramsey rule or the TVC;
  2. $z_1 = (k(0) - \overline{k})/v_{11}$.

Therefore, from Eq. (29), we can analytically obtain the optimal growth path as follows:

$$k(t)-\overline{k} = (k(0)-\overline{k})\exp(\xi_2 t), \quad c(t)-\overline{c} = (v_{21}/v_{11})(k(0)-\overline{k})\exp(\xi_2 t). \tag{31}$$

Characterization of Competitive Equilibrium

Decentralized Economy

- Now suppose that the economy is decentralized, rather than centrally planned.
- There are
  1. Two factor markets, one for labor and one for capital services; $w(t)$: the wage rate, $R(t)$: the rental price of capital.
  2. a good market (numeraire),
  3. a debt market in which households can borrow and lend.

Firms

- There are many identical firms, each with the same technology. Firms rent capital and hire workers to produce output.

- A representative firm's profit maximization problem:

$$\max_{K(t),L(t)} F(K(t), L(t)) - R(t)K(t) - w(t)L(t).$$

Since $F(K, L) = f(k)L$, the above problem can be converted to

$$\max_{k(t),L(t)} [f(k(t)) - R(t)k(t) - w(t)]L(t)$$

- Competitive factor markets then imply:

$$R(t) = f'(k(t)), \tag{32}$$

and

$$w(t) = f(k(t)) - k(t)f'(k(t)). \tag{33}$$

Households

- Let $\mathcal{A}(t)$ denote asset holdings of the representative household at t. Then,

$$\dot{\mathcal{A}}(t) = r(t)\mathcal{A}(t) + w(t)L(t) - c(t)L(t), \tag{34}$$

  (∗) $r(t)$: the risk-free market flow rate of return on assets.

  (∗) We will discuss the relationship between r and R soon later.

- Define a as follows:

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

- Then, we obtain the flow budget equation in per captia terms:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t). \tag{35}$$

Budget Constraint and No-Ponzi-Game Condition

- Household assets can consist of capital stock, $K(t)$, which they rent to firms and individual bonds, $B_p(t)$ which is circulated among households.

- In per capita term,

$$a(t) = k(t) + b_p(t).$$

  Note that $b_p(t) \equiv B_p(t)/L(t)$ can be negative, also $a(t)$ can be negative.

- So the differential equation

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t),$$

  is a constraint, but not sufficient as a proper budget constraint unless we impose a lower bound on $a(t)$.

Budget Constraint and No-Ponzi-Game Condition

- In an infinte-horizon economy, the following no-Ponzi-game (NPG) condition is imposed:

$$\lim_{t\to\infty}\left[a(t)\exp\left(-\int_0^t (r(s)-n)ds\right)\right]\geq 0. \tag{36}$$

  This equation means that the discounted value of assets must not be negative in the infinite future.

- Why must we impose the equation?
- To see why, let us derive the intertemporal budget constraint.

Budget Constraint and No-Ponzi-Game Condition

- Multiply both sides of (36) by $\exp(-\int_0^t (r(s)-n)ds)$ and rearranging terms:

$$[\dot{a}(t)-(r(t)-n)a(t)]\exp\left(-\int_0^t (r(s)-n)ds\right) = [w(t)-c(t)]\exp\left(-\int_0^t (r(s)-n)ds\right).$$

- Note that the left hand side (LHS) satisfies

$$[\dot{a}(t)-(r(t)-n)a(t)]\exp\left(-\int_0^t (r(s)-n)ds\right) = \frac{d\left[a(t)\exp\left(-\int_0^t (r(s)-n)ds\right)\right]}{dt}.$$

Budget Constraint and No-Ponzi-Game Condition

- Then,

$$\frac{d\left[a(t)\exp\left(-\int_0^t (r(s)-n)ds\right)\right]}{dt} = [w(t)-c(t)]\exp\left(-\int_0^t (r(s)-n)ds\right).$$

- Integrating the above equation from 0 to an arbitrary T:

$$a(T)\exp\left(-\int_0^T (r(s)-n)ds\right) - a(0) = \int_0^T [w(t)-c(t)]\exp\left(-\int_0^t (r(s)-n)ds\right)dt.$$

This is the intertemporal budget constraint.

- Rearranging terms:

$$a(0) - a(T)\exp\left(-\int_0^T (r(s)-n)ds\right) + \int_0^T w(t)\exp\left(-\int_0^t (r(s)-n)ds\right)dt$$
$$= \int_0^T c(t)\exp\left(-\int_0^t (r(s)-n)ds\right)dt.$$

  - LHS is (life-time) income, and RHS is life-time expenditure.
  - If the 2nd term in LHS (red-colored) could be negative, the household would choose any path of $c(t)$ by accumulating her debts: the maximization problem becomes a trivial one.
  - NPG condition must be imposed to prohibit such a situation: $a(T)\exp\left(-\int_0^T (r(s)-n)ds\right) \geq 0$.

- Taking a limit of $T \to \infty$, we obtain (36).

Households' Intertemporal Optimization

- Then, the representative household's utility maximization problem is given by:

$$\max_{c(t)} \quad U(0) = \int_0^\infty \exp(-(\rho - n)t)u(c(t))dt$$

$$\text{subject to} \quad \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t),$$

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - n)ds\right) \geq 0,$$

with taking the following variables as given:

1. a(0): The initial condition;
2. The times paths of $r(t)$ and $w(t)$, which implies that each household is a price taker.

- 1st constraint is the flow budget equation, while 2nd constraint is a kind of solvency constraint.

Households' Intertemporal Optimization

- The current-value Hamiltonian is

$$\hat{H}(a, c, \mu) = u(c) + \mu[(r-n)a + w - c].$$

- Then, the conditions for utility maximization is

$$u'(c) = \mu, \tag{37}$$

$$\dot{\mu} = (\rho - r)\mu, \tag{38}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)a(t) = 0. \tag{39}$$

Euler Equation and Transversality Condition

- From (37) and (38),

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}\Big(r(t) - \rho\Big). \tag{40}$$

- From (38) and (39),

$$\lim_{t\to\infty} a(t) \exp\left(-\int_0^t (r(s) - n)ds\right) = 0. \tag{41}$$

$(*)$That is, the TVC means that the present value of assets at the indefinitely future date becomes zero.

$(*)$ the NPG condition is the constraint which prevents households from strategically defaulting, whereas TVC is the condition for optimality such that NPG condition is satisfied with equality.

Summary of Household Bahavior

- From Household's utility maximization, we get the following three conditions:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t),$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))}(r(t) - \rho),$$

$$\lim_{t \to \infty} a(t) \exp\left(-\int_0^t (r(s) - n)ds\right) = 0.$$

Market Equilibrium Conditions

- Recall that households' aggregate portfolio is $\mathcal{A}(t) = K(t) + B_p(t)$, where $K(t)$ is individual bonds.
- Note that $B(t) = 0$ since the bonds are in zero net supply. This implies $\mathcal{A}(t) = K(t)$.
- Then, asset market-clearing condition in *per capita terms* is given by

$$a(t) = k(t). \tag{42}$$

Market Equilibrium Conditions

- Assume that capital stock depreciates at the rate $\delta > 0$.
- Since $a(t) = k(t)$, the market rate of return on assets, $r(t)$, satisfies

$$r(t) = R(t) - \delta. \tag{43}$$

Definition of Equilibrium

Definition

A competitive equilibrium of the model consists of time paths of $[w(t), R(t), a(t), c(t), k(t), r(t)]_{t=0}^{\infty}$ such that:

1. $[w(t), R(t)]_{t=0}^{\infty}$ are given by firms' profit maximization, (32) and (33);
2. $[a(t), c(t)]_{t=0}^{\infty}$ are given by households' utility maximization (35), (40), together with the initial condition $a(0)$ and the transversality condition (41);
3. $[k(t)]_{t=0}^{\infty}$ is given by the market clearing condition for assets (42);
4. $[r(t)]_{t=0}^{\infty}$ is given by (43)

Reduced Dynamics

- Equilibrium is characterized by time paths of 6 variables $[w(t), R(t), a(t), c(t), k(t), r(t)]_{t=0}^{\infty}$ with $a(0)(= k(0))$ given.

- However, once the time paths of $[k(t), c(t)]_{t=0}^{\infty}$ are determined, the other variables are accordingly determined.
  I will show this using the next two slides.

Reduced Dynamics

- Using the asset market equilibrium condition (42) and (43), we can arrange the budget equation (35).

$$\begin{aligned} & \dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) \\ \Rightarrow \quad & \dot{k}(t) = (R(t) - \delta - n)k(t) + w(t) - c(t). \end{aligned}$$

- From the firms' F.O.Cs (32) and (33),

$$\begin{aligned} R(t)k(t) + w(t) &= \underbrace{f'(k(t))}_{=R(t)} k(t) + \underbrace{f(k(t)) - k(t)f'(k(t))}_{=w(t)} \\ &= f(k(t)). \end{aligned}$$

- Substituting this result into the above equation, we obtain the goods market equilibrium (the resource constraint):

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t). \tag{44}$$

Reduced Dynamics

- By the same way we can arrange the Euler equation (40) as follows:

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\varepsilon_u(c(t))}(r(t)-\rho) \\ &= \frac{1}{\varepsilon_u(c(t))}(f'(k(t))-\delta-\rho). \qquad (45)\end{aligned}$$

- Finally, substituting the market-clearing condition for assets, (42) into TVC,

$$\lim_{t\to\infty} k(t)\exp\left(-\int_0^t (f'(k(s))-\delta-n)ds\right)=0. \qquad (46)$$

Social Planning and the Market Economy

- Equations (44), (45) and (46) characterize the behavior of the (competitive) market economy.
- Note that they are identical to (11), (16) and (18), which characterizes the optimal growth path chosen by a social planner.
- Thus, the dynamic behavior of the competitive market economy is the same as that of the centrally planned one.

The Government in the Decentralized Economy

The Government in Decentralized Economy

- We now introduce the government into the model.
- The covered topics are
  1. Effects of government spending;
  2. Effects of debt financing; and
  3. Effects of taxations

Effects of Government Spending

- Suppose that the government
  1. Expends $G(t)$ units of the final good (wasteful government spending);
  2. Levies lump-sum taxes $\tau(t)$ on each household to finance the expenditure.

- Therefore the government's budget constraint is

$$\tau(t)L(t) = G(t) \Leftrightarrow \tau(t) = g(t). \tag{47}$$

- The household's flow budget constraint now becomes

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t) - \tau(t). \tag{48}$$

Dynamic System

- The dynamics of $c(t)$:

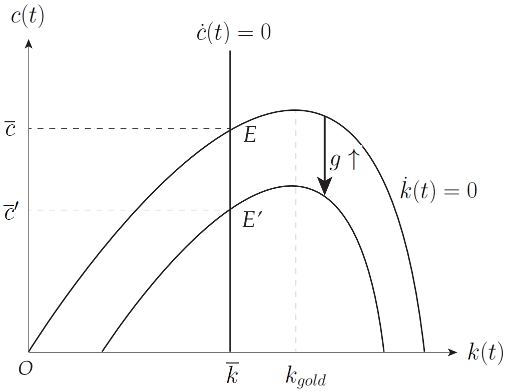
$$\varepsilon_u(c(t))\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho.$$

- On the other hand, the dynamics of $k(t)$ now becomes

$$\dot{k}(t) = f(k(t)) - (n+\delta)k(t) - c(t) - g(t).$$

Effects of Government Spending

- For simplicity, we assume that $g(t) = g > 0 \forall t \geq 0$.



- In the steady state, the government spending completely crowds out private consumption, *but has no effect on the capital stock!*

Effects of Debt Financing

- Now consider the case that the government is allowed to borrow, instead of financing its spending only through taxes.
  - Let $B_g(t) \geq 0$ denote the level of the government debt (国債残高).
- The government's budget constraint is now given by

$$\dot{B}_g(t) = r(t)B_g(t) + G(t) - \tau(t)L(t). \tag{49}$$

  - If $\dot{B}_g(t)$ is positive, the government runs a budget deficit (財政赤字).
  - If $(G(t) - \tau(t))L(t)$ is positive, the government runs a primary deficit.

The Government's Intertemporal Budget Constraint

- Let $b_g(t)$ denote the level per capita government debt.

$\Downarrow$

(49) is rewritten as

$$\dot{b}_g(t) = (r(t) - n)b_g(t) + g(t) - \tau(t).$$

- Integrating the above equation from zero to infinity,

$$b_g(0) = \int_0^\infty (\tau(t) - g(t)) \exp\left(-\int_0^t (r(s) - n)ds\right) dt + \lim_{t\to\infty} b_g(t) \exp\left(-\int_0^t (r(s) - n)ds\right). \tag{50}$$

The Government's Intertemporal Budget Constraint

- The NPG condition of the government is

$$\lim_{t\to\infty} b_g(t)\exp\left(-\int_0^t (r(s)-n)ds\right)=0.$$

- This leads the following result:

$$b_g(0)=\int_0^{\infty}(\tau(t)-g(t))\exp\left(-\int_0^t (r(s)-n)ds\right)dt. \tag{51}$$

(∗) (51) は政府の異時点間予算制約式 (intertemporal budget constraint) と呼ばれる.

Ricardian Neutrality

- The household's portfolio now becomes

$$a(t) = k(t) + b_p(t) + b_g(t). \tag{52}$$

- Then, the household's intertemporal budget constraint is given by

$$\int_0^\infty c(t) \exp\left(-\int_0^t (r(s) - n)ds\right) dt$$
$$= k(0) + b_p(0) + b_g(0) + \int_0^\infty (w(t) - \tau(t)) \exp\left(-\int_0^t (r(s) - n)ds\right) dt. \tag{53}$$

Ricardian Neutrality

- Then, using Eq. (51) and (53),

$$\int_0^\infty c(t)\exp\left(-\int_0^t (r(s)-n)ds\right)dt$$
$$+k(0)+b_p(0)+\int_0^\infty (w(t)-g(t))\exp\left(-\int_0^t (r(s)-n)ds\right)dt.$$

$\Downarrow$

- Key results:
    1. Neither taxes nor government debt appear in the budget constraint; and
    2. Only government spending matters.

Ricardian Neutrality

This result is summarized as follows:

Theorem

For any given path of $g(t)$, the method of finance, whether distortionless taxation or budget deficit has no effect on equilibrium allocation.

This theorem is called the Ricardian equivalence theorem (リカードの等価定理).

Effects of Taxations

- Finally, we consider the effects of taxations.

- We now consider the following household' budget constraint

$$\dot{a}(t) = [(1-\tau_r)r(t) - n]a(t) + (1-\tau_w)w(t) - (1+\tau_c)c(t) - \tau(t), \qquad (54)$$

  where
  - $\tau_r \in [0,1)$, $\tau_w \in [0,1)$, $\tau_c \geq 0$ are respectively the rates of interest income, wage income, and consumption.

- The Euler equation now becomes

$$\varepsilon_u(c(t))\frac{\dot{c}(t)}{c(t)} = (1-\tau_r)r(t) - \rho. \qquad (55)$$

  $\Rightarrow$ Only capital income taxation is distortionary !

Effects of Taxations

- To focus on the effects of taxation, we assume that the government does not issue the public bond and $g(t) = 0$.

- Namely, the government's budget constraint is

$$\tau_r r(t)k(t) + \tau_w w(t) + \tau_c c(t) + \tau(t) = 0,$$

which implies $\tau(t) < 0$.

($*$) This means that the tax revenue $\tau_r r(t)k(t) + \tau_w w(t) + \tau_c c(t)$ is subsidized to the household in a lump-sum manner.

$\Downarrow$

- From the household's budget constraint (53) and the firm's condition, we obtain $\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$

- With $r(t) = f'(k(t)) - \delta$, the Euler equation with $\dot{c}(t) = 0$ is given by

$$f'(k^*) = \frac{\rho}{1-\tau_r} + \delta$$

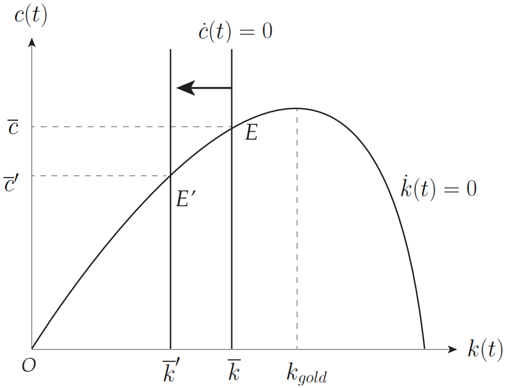


Figure: Effets of capital income taxation

Taking a Stock

- A contribution of the RCK model is that it opens the black box of savings and capital accumulation by specifying preferences and technologies.
  - In a Solow-Swan growth model, the saving rate was taken as exogenous.
  - In the RCK model, we can link the saving to preferences, technology and prices.
- Another major contribution of the model is that because preferences are explicitly specified, optimal path and equilibrium paths can be compared.
  - Optimal growth paths is a unique converging path to a modified golden rule, not the golden rule
    $\Rightarrow$ steady-state consumption is NOT a proxy of welfare.
  - Competitive equilibrium path corresponds to the optimal growth one.

補論：主体の同質性の仮定について

- 前章の Ramsey-Cass-Koopmans モデルの特徴の 1 つ：
  「同質的な家計主体が，無限視野の計画期間を持つ」
- RCK モデルは別名「代表的個人モデル」
  - 些か非現実的？
  - より rich な疑問は「どのような条件下で正当化できる？」

$\Downarrow$

- 異質である，という仮定から出発して，前章の結果はどこまで頑健だろうか？

主体の同質性の仮定について

- 簡単化のため人口成長はなく，サイズは L で固定
- 個人 $i \in [0, L]$ の問題 ：

$$\begin{aligned} \max \quad & U_i = \int_0^\infty u_i(c_i(t)) \exp(-\rho_i t) dt \\ \text{s.t.} \quad & \dot{a}_i(t) = r(t) a_i(t) + w_i(t) - c_i(t). \end{aligned}$$

- オイラー方程式：

$$-\frac{c_i(t) u_i''(c_i(t))}{u_i'(c_i(t))} \frac{\dot{c}_i(t)}{c_i(t)} = r(t) - \rho_i \tag{56}$$

主体の同質性の仮定について

- x_t を変数 x_{it} の平均 ($x_t = \int_0^L x_{it} di/L$) とすると,

$$\dot{a}(t) = (1/L) \int_0^L \dot{a}_{it} di = r(t)a(t) + w(t) - c(t)$$

- 一方，オイラー方程式より

$$\dot{c}(t) = (1/L) \int_0^L \dot{c}_i(t) di = (1/L) \int_0^L -\frac{u_i'(c_i(t))}{u_i''(c_i(t))}(r(t) - \rho_i) di$$

ここで,

- $-u_i'(\cdot)/u_i''(\cdot)$ は絶対的危険回避度 (ARA) の逆数

主体の同質性の仮定について

- 任意の i に関して $\rho_i = \rho$ としよう.
- $-u_i'(\cdot)/u_i''(\cdot)$ が消費に関して線形，かつ

$$-u_i'(\cdot)/u_i''(\cdot) = A_i + Bc_i \tag{57}$$

のように微係数が個人間で同じ
$\Downarrow$

$$\dot{c}(t) = (A + Bc(t))(r(t) - \rho), \quad A \equiv \int A_i di/L.$$

マクロ変数が労働所得や資産所得の異質性に依存しない (集計特性).

主体の同質性の仮定について

- 以下の条件が成立するとき, 異質な主体からなるマクロ変数の動きは，ある代表的個人の変数の動き

  1. 選好が主体間で同じ，かつ集計特性を満たす
  2. すべての主体が同じ利子率に面している

  $\Uparrow$
  この背景には，より重要な仮定「全ての主体が等しい機会で市場に参加できる」がある