Advanced Macroeconomics

(Department of Social Engineering, Spring FY2015)

Dynamic Optimization in Continuous Time

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Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures incl. today)
  2. The Ramsey-Cass-Koopmans Model (3 lectures)
  3. Endogenous Growth Models (2 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic Dynamic Programming (3 lectures)

Plan of Lecutres in the Part of "Dynamic Optimziation"

- April, 8 (Wed): An introduction to dynamic optimization
- April, 15 (Wed) (Today): Infinite-horizon dynamic programming
- April, 22 (Wed): Discrete dynamical system
- April, 30 (Thu): Continuous-time optimal control

Introduction

- So far, we have considered dynamic optimization in discrete time.
- More specifically,
  1. If you face a finite-horizon problem (ex. cake-eating problem on 4/8 slides),
     1. Applications of mathematical programming;
        or
     2. Solve the Bellman equation by "backward induction,"
  2. If you face an infinite-horizon problem $\Rightarrow$ Infinite-horizon DP (4/15 slides)

     The optimal path is given by
     - ★ the policy function (if the function V in the Bellman Eq. is explicitly found);
       or
     - ★ the Euler equation and the TVC (If V is differentiable).

Introduction

$\Downarrow$

1. Then, using the knowledge of discrete dynamical system, you can examine the characteristics of the optimal path (4/22 slides).

- What are the characteristics?

  $\Downarrow$

  1. Existence of the steady states
  2. Uniqueness of —
  3. (Local) Stability of —

- You must solve the linearized dynamical system of difference equations.

Introduction

- Dynamic optimization in continuous time is also useful tool for macroeconomics, and other areas of dynamic economic analysis.
- Therefore, the basic tools of dynamic optimization in continuous time are introduced.

Continuous-time Cake-eating problem (finite-horizon)

- Consider the following problem, which is a continuous-time counterpart of the cake-eating problem on 4/8 slides.

$$\max_{c:[0,T]\to\mathbb{R}_+} \quad U[c] = \int_0^T \exp(-\rho t) u(c(t)) dt,$$
$$\text{s.t.} \quad \int_0^T c(t)dt \leq W,$$
$$c(t) \geq 0.$$

- Notation:
  - $U[c]$ is the life-time utility functional;

    ($*$) Briefly speaking, a functional (汎関数) is a function of functions. However, in economics, U is often simply called the life-time utility function.

  - In continuous-time problems, the one-period utility function u is called the instantaneous utility function (瞬時的効用関数).

Continuous-time Cake-eating problem (finite-horizon)

- As well as the discrete-time cake-eating problem on 4/8 slides, we assume

$$u'(c) > 0,\ u''(c) < 0,\ \lim_{c \to 0} u'(c) = \infty.$$

  - From the 1st assumption, $\int_0^T c(t)dt \leq W$ binds; and
  - From the 3rd assumption, $c(t) \geq 0$ never binds.

- Then, the above problem is simplified to

$$\max_{c:[0,T]\to\mathbb{R}_+} \quad U[c] = \int_0^T \exp(-\rho t)u(c(t))dt, \tag{P0}$$
$$\text{s.t.} \quad \int_0^T c(t)dt = W.$$

Variational Problem

- This type of problem is called the variational problem (変分問題), and basically solved by the calculus of variation (変分法).

Proposition (Necessary Condition of Maximization)

Let $c^* : [0, T] \to \mathbb{R}_{++}$ is the interior solution to the cake-eating problem. Then, there exists $\lambda \geq 0$ such that $c^*(t)$ satisfies

$$e^{-\rho t} u'(c^*(t)) = \lambda \forall t \in [0, T]. \tag{1}$$

Sketch of Proof

- Let $\gamma : [0, T] \to \mathbb{R}$ denote a continuous function of time, and let $\varepsilon \in \mathbb{R}$ be a real number.

- Let us define the valuation of $c^*(t)$ by

$$c(t, \varepsilon) = c^*(t) + \varepsilon\gamma(t).$$

Assumption

$c(t, \varepsilon)$ *is feasible: i.e.,* $\int_0^T c(t, \varepsilon)dt \leq W$.

Obviously,

$$c(t, 0) = c^*(t) \forall t \in [0, T].$$

Sketch of Proof

- Consider the following Lagrangian:

$$L(\varepsilon) = \int_0^T e^{-\rho t} u(c(t,\varepsilon))dt + \lambda \left(W - \int_0^T c(t,\varepsilon)dt \right).$$

- Differentiating L w.r.t. ε,

$$\begin{aligned} L_\varepsilon(\varepsilon)(\equiv dL(\varepsilon)/d\varepsilon) &= \int_0^T \left(e^{-\rho t} u'(c(t,\varepsilon)) - \lambda\right) \frac{\partial c(t,\varepsilon)}{\partial \varepsilon} dt \\ &= \int_0^T \left(e^{-\rho t} u'(c(t,\varepsilon)) - \lambda\right) \gamma(t) dt. \end{aligned}$$

- Evaluate this at $\varepsilon = 0$:

$$L_\varepsilon(0) = \int_0^T \left(e^{-\rho t} u'(c^*(t)) - \lambda\right) \gamma(t) dt. \tag{2}$$

Sketch of Proof

- If there exist some $\gamma(t)$'s such that $L_\varepsilon(0) > 0$, we can obtain higher utility by deviating from $c^*(t)$.
- This contradicts the assumption that $c^*(t)$ is optimal. Then,

$$L_\varepsilon(0) = 0 \forall \gamma(t),$$

which results in

$$\exp(-\rho t)u'(c^*(t)) = \lambda \forall t \in [0, T].$$

$\square$

Note

- (1) corresponds to the first-order-condition of the problem*as if we could independently choose $c(t)$ at each point in time*.
  $\Updownarrow$
  We have the same condition as the cake-eating problem in discrete time.

- Then, can we use the method on Lagrangian multipliers, or the Kuhn-Tucker condition in a class of continuous-time models ?

  To answer this question, we have to examine the sufficiency of the F. O. C.

Variational Problem

Proposition (Sufficient Condition)

Suppose that $u'' < 0$*, and that* $c^* : [0, T] \to \mathbb{R}_{++}$ *and* $\lambda \geq 0$ *satisfy*

$$u'(c^*(t)) = \lambda \forall t \in [0, T],$$

$$\int_0^T c^*(t)dt = W.$$

Then, $c^*(t)$ *gives the unique interior solution to the problem.*

Proof.

Exercise. □

Finite-horizon Optimal Control in Continuous Time

- The canonical continuous-time optimization problem is given by:

$$\begin{aligned} \max_{c:[0,T]\to\mathbb{R}_+} \quad & J[x,c] = \int_0^T F(x(t),c(t),t)dt \\ \text{subject to} \quad & \dot{x}(t) \equiv \frac{dx(t)}{dt} = g(x(t),c(t)) \quad 0 \le t \le T, \\ & x(T) \ge 0, \quad x(0) \text{ given.} \end{aligned} \tag{P}$$

  where
  - J is the *objective functional*, while $F(\cdot)$ is the *one-period return function*.
  - $c : [0,T] \to C \subseteq \mathbb{R}_+$ is called the *control variable*.
  - On the other hand, $x : [0,T] \to X \subseteq \mathbb{R}_+$, is called the *state variable*. This kind of variables is determined only indirectly thorough the *transition equation*.

Finite-horizon Optimal Control in Continuous Time

- Three approaches:
  1. Calculus of Variations;
  2. Optimal Control (based on Pontryagin's maximum principle); and
  3. Continuous-time Dynamic Programming (based on Bellman's principle of optimality).
- We will mainly focus on the second approach.

Preliminary

- Define the following functional $\hat{L}[x,c]$:

$$\hat{L}[x,c] = \int_0^T \Big[F(x(t),c(t),t)dt + \lambda(t)\Big(g(x(t),c(t)) - \dot{x}(t)\Big)\Big]dt$$

where

  - $\lambda(t) \geq 0$ is called the costate variable (共役変数) or the adjoint variable (随伴変数).

(∗) 簡単にいうと, $\hat{L}[x,c]$ は終点条件 $x(T) \geq 0$ を一旦忘れた際のラグランジュ(汎) 関数.

Preliminary

- We can arrange $\hat{L}[x,c]$ as

$$\hat{L}[x,c] = \int_0^T \Big[F(x(t),c(t),t)dt + \lambda(t)g(x(t),c(t))\Big]dt - \int_0^T \lambda(t)\dot{x}(t)dt.$$

Furthermore,

1. 1st term of the RHS is simplified when we define the following function:

$$H(x,c,\lambda,t) \equiv F(x,c,t) + \lambda g(x,c). \tag{3}$$

H is called the Hamiltonian.

2. 2nd term of the RHS is rewritten as

$$\int_0^T \lambda(t)\dot{x}(t)dt = \lambda(T)x(T) - \lambda(0)x(0) - \int_0^T x(t)\dot{\lambda}(t)dt = 0. \tag{4}$$

Preliminary

- In sum,

$$\hat{L}[x,c] = \int_0^T \Big[H(x(t),c(t),\lambda(t),t) + x(t)\dot{\lambda}(t)\Big]dt + \lambda(0)x(0) - \lambda(T)x(T).$$

Assumption

Both of F and g are continuously differentiable.

$\Rightarrow$ The Hamiltonian H becomes continuously differentiable.

- To simplify notation, hereafter we let H_x, H_c and H_λ denote the partial derivatives of H with respect to x, c and λ, respectively:

$$H_j(x,c,\lambda,t) = \frac{\partial H(x,c,\lambda,t)}{\partial j}, \quad j = x, c, \lambda.$$

Pontryagin's Maximum Principle

Theorem (**Simplified Maximum Principle**)

Consider the problem (P). Suppose that this problem has the interior solution $(x^*(t), c^*(t))$. *Then, there exists* $\lambda(t) \geq 0$ *such that* $x^*(t)$ *and* $c^*(t)$ *satisfy the following conditions:*

$$H_c(x^*(t), c^*(t), \lambda(t), t) = 0 \quad \forall t \in [0, T], \tag{5}$$

$$\dot{\lambda}(t) = -H_x(x^*(t), c^*(t), \lambda(t), t) \quad \forall t \in [0, T], \tag{6}$$

$$\dot{x}(t) = H_\lambda(x^*(t), c^*(t), \lambda(t), t) \quad \forall t \in [0, T], \tag{7}$$

and the following terminal condition:

$$x^*(T) \geq 0, \ \lambda(T) \geq 0, \ \lambda(T)x^*(T) = 0. \tag{8}$$

In (8), $\lambda(T)x^*(T) = 0$ is called the transversality condition (横断性条件).

Sketch of Proof

- Let us define the valuation of $c^*(t)$ by

$$c(t, \varepsilon) = c^*(t) + \varepsilon\gamma(t).$$

- Let us also define $x(t, \varepsilon)$ as the path of the state variable corresponding to the path of control variable $c(t, \varepsilon)$. We assume that also $x(t, \varepsilon)$ is feasible: i.e., $x(t, \varepsilon)$ satisfies

$$\dot{x}(t, \varepsilon) \left(\equiv \frac{dx(t, \varepsilon)}{dt} \right) = g(x(t, \varepsilon), c(t, \varepsilon)).$$

  Since the initial state is historically given, $x(0, \varepsilon) = x(0)$ must hold. Then,

$$x(t, 0) = x^*(t) \forall t \in [0, T].$$

Sketch of Proof

- Then, define $\mathcal{J}(\varepsilon)$ by

$$\mathcal{J}(\varepsilon) = \hat{L}[x(t,\varepsilon), c(t,\varepsilon)]$$
$$= \int_0^T \Big[H\big(x(t,\varepsilon), c(t,\varepsilon), \lambda(t), t\big) + x(t,\varepsilon)\dot{\lambda}(t)\Big]dt + \lambda(0)x(0) - \lambda(T)x(T,$$

- Define the $\mathcal{L}(\varepsilon)$ by

$$\mathcal{L}(\varepsilon) = \mathcal{J}(\varepsilon) + \zeta x(T,\varepsilon)$$

(∗) 簡単にいうと, $\mathcal{L}$ は終点条件 $x(T) \geq 0$ を加味した上でのラグランジュ関数.

Sketch of Proof

- 後は cake-eating problem のときと同じ：

$$\mathcal{L}_\varepsilon(0) = 0 \Leftrightarrow \mathcal{J}_\varepsilon(0) + \zeta \frac{\partial x(T,0)}{\partial \varepsilon} = 0$$
$$\Leftrightarrow \int_0^T \left[H_c^* \gamma(t) + (H_x^* + \dot{\lambda}) \frac{\partial x(t,0)}{\partial \epsilon} \right] dt + (\zeta - \lambda(T)) \frac{\partial x(T,0)}{\partial \varepsilon}, \tag{9}$$

and

$$x^*(T) \geq 0, \;\; \zeta \geq 0, \zeta x^*(T) = 0. \tag{10}$$

- Then, (9), (10) and the transition equation provide (5)–(8).

Sufficiency

- When are the necessary condition of optimality both necessary and sufficient?

Theorem (**Mangasarian's Sufficiency Theorem**)

Consider the problem (P). Suppose that

1. $(x^*(t), c^*(t))$ *and* $\lambda(t)$ *satisfy the conditions (5)–(8).*
2. *Both of* F *and* g *satisfy Assumption 2, and*
3. *They are concave with respect to* (x, c) *for all* $t \in [0, T]$.

Then $(x^*(t), c^*(t))$ *solve the problem (P).*

Proof.

Given in the supplementary material. $\square$

Discounted Problem and Current-value Hamiltonian

- For many problems in economics, future values of returns are discounted:

$$F(x, c, t) = \exp(-\rho t) f(x, c),$$

where
  - $\rho > 0$ is called the discount rate (割引率).

- Then, the problem (P) is now given by

$$\begin{aligned} \max \quad & J = \int_0^T \exp(-\rho t) f(x(t), c(t)) dt \\ \text{subject to} \quad & \dot{x}(t) = g(x(t), c(t)) \quad 0 \leq t \leq T, \\ & x(T) \geq 0, \quad x(0) \text{ given.} \end{aligned} \tag{P'}$$

Discounted Problem and Current-value Hamiltonian

- 単に $F(x,c,t)$ の “t” の影響の仕方を特定化しただけなので，問題の本質は変わらない. ⇒The Hamiltonian is given by

$$H(x(t),c(t),\lambda(t),t) = \exp(-\rho t)f(x(t),c(t)) + \lambda(t)g(x(t),c(t)).$$

- Rather, we can simplify the Hamiltonian thanks to the time-discounting. Define the following new variable:

$$\mu(t) = \exp(\rho t)\lambda(t),$$

and the new function:

$$\hat{H}(x(t),c(t),\mu(t)) = (x(t),c(t)) + \mu(t)g(x(t),c(t)).$$

$\hat{H}$ is called the current-value Hamiltonian (当該価値ハミルトニアン).[1]

[1] On the other hand, H is called the present-value Hamiltonian (現在価値ハミルトニアン).

Discounted Problem and Current-value Hamiltonian

- Using this current-value Hamiltonian, we can rewrite the conditions (5)–(8) more simply:

$$\hat{H}_c(x^*, c^*, \mu) = 0 \Leftrightarrow f_c(x^*, c^*) + \mu g_c(x^*, c^*) = 0, \tag{11}$$

$$\dot{\mu} = \rho\mu - \hat{H}_x(x^*, c^*, \mu) \Leftrightarrow \dot{\mu} = \rho\mu - (f_x(x^*, c^*) + \mu g_x(x^*, c^*)) = 0, \tag{12}$$

$$\dot{x} = \hat{H}_\mu(x^*, c^*, \mu) \Leftrightarrow \dot{x} = g(x, c), \tag{13}$$

$$\mu(T)\exp(-\rho T) \geq 0,\ \ x^*(T) \geq 0,\ \ \mu(T)x^*(T)\exp(-\rho T) = 0. \tag{14}$$

Summary of Procedure to Solve the Problem

- So, when you encounter a discounted optimization problem in continuous time, use the following cookbook procedure.
  1. Construct the current-value Hamiltonian:
  $$\hat{H}(x, c, \mu) = f(x, c) + \mu g(x, c).$$
  2. Take the derivative of Hamiltonian with respect to c and set it equal to 0: i.e., $\hat{H}_c = 0$.
  3. Take the derivative of Hamiltonian with respect to x, and set it equal to $\rho\mu - \dot{\mu}$, i. e.,
  $$\dot{\mu} = \rho\mu - \hat{H}_x. \tag{15}$$
  4. Derive the TVC from the complementary slackness condition for $x(T) \geq 0$.