Advanced Macroeconomics

(Department of Social Engineering, Spring FY2015)

Dynamic Optimization in Discrete Time (3)
Discrete Dynamical System

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Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures incl. today)
  2. The Ramsey-Cass-Koopmans Model (3 lectures)
  3. Endogenous Growth Models (2 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic Dynamic Programming (3 lectures)

Plan of Lecutres in the Part of "Dynamic Optimziation"

- April, 8 (Wed): An introduction to dynamic optimization
- April, 15 (Wed) : Infinite-horizon dynamic programming
- April, 22 (Wed) (Today): Discrete dynamical system
- April, 30 (Thu): Continuous-time optimal control
  (∗) Note the day of week, not the same as usual.

Summary of Infinite-horizon DP

- The value function (価値関数) of the original problem:

$$V^*(x) = \max_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}).$$

- Principle of Optimality (最適性の原理)
  1. $V^*(x)$ satisfies the Bellman equation (ベルマン方程式):

$$V(x) = \max_{x'} \{f(x, x') + \beta V(x, x')\}. \tag{1}$$

  2. If V is a solution to the Bellman equation and if it satisfies

$$\lim_{t \to \infty} \beta^t V(x_t) = 0 \ \forall x_0 \in X, \{x_t\}_{t=1}^{\infty} \in \Pi(x_0), \tag{2}$$

  then, $V(x) = V^*(x)$.

Summary of Infinite-horizon DP

- If we can explicitly obtain the value function, $V^*(x)$, we can accordingly obtain the optimal path $\{x_t^*\}_{t=1}^{\infty}$ from

$$\forall x_0 \in X,\ x_{t+1}^* = h(x_t^*) \quad t = 0, 1, 2, \ldots. \tag{3}$$

where $h : X \to X$ is the policy function (政策関数).

- If we can not, but if we know f and V are differentiable,

$$\forall x_0 \in X,\ f_2(x_t^*, x_{t+1}^*) + \beta f_1(x_{t+1}^*, x_{t+2}^*) = 0 \quad t = 0, 1, 2, \ldots, \tag{4}$$

$$\lim_{t\to\infty} \beta^t f_1(x_t^*, x_{t+1}^*) x_t^* = 0. \tag{5}$$

Discrete Dynamical System

- A discrete dynamical system [離散力学系 (or 離散動学系)] is the system, where the dynamics of variables are described by the system of difference equations (連立差分方程式).

- (3) is a first-order difference equation of x_t, whereas the Euler equation (4) is a second-order difference equation of that.

- Let $y_t = x_{t+1}$. Then, the Euler equation (4) is expressed as

$$f_2(x_t^*, y_t^*) + \beta f_1(y_t^*, y_{t+1}^*) = 0, \tag{6}$$

$$x_{t+1}^* = y_t^*. \tag{7}$$

⇒ Therefore, we hereafter consider a dynamical system given by the system of first-order difference equation.

Discrete Dynamical System

- Let
  - $\boldsymbol{x}_t = (x_{1,t}, x_{2,t}, \dots, x_{N,t}) \in X \subseteq \mathbb{R}^N$ denote the N dimensional vector; and
  - $g : X \to X$ denote a function.

Definition

A discrete dynamical system is given by a triplet $(\mathbb{Z}_+, X, g)$. In the system, the dynamics of $\boldsymbol{x}_t$ is given by

$$\boldsymbol{x}_{t+1} = g(\boldsymbol{x}_t) \quad t = 0, 1, 2, \dots$$

($*$) When g does not depend on time as above, the system is called autonomous (自励的), while called non-autonomous (非自励的) if it does.

Autonomous- and Non-autonomous systems

- Example: In the Solow growth model with production technology given by $y = A_t k^\alpha$ $(\alpha \in (0,1))$, the dynamics of per-capita physical capital is

$$k_{t+1} = sy_t = sA_t k_t^\alpha, \tag{8}$$

where $s \in (0,1)$ is the constant saving rate.

  - If there is no technological progress, A_t is constant over time (i.e., $A_t = A > 0$).
    $\Rightarrow$ (8) governs the autonomous system of k_t.
  - If A_t evolves according to

$$A_{t+1} = (1+g)A_t \text{ where } g > 0, \tag{9}$$

then, (8) is no longer autonomous by itself.
$\Rightarrow$ (8) and (9) jointly constitute the autonomous system of (k_t, A_t).

Steady State

- Hereafter, we focus on
  1. An autonomous system, $\boldsymbol{x}_{t+1} = g(\boldsymbol{x}_t)$; and
  2. g is continuous.

Definition

$\overline{\boldsymbol{x}} \in X$ is called the steady state (定常状態) or stationary state (〃) if $\overline{\boldsymbol{x}}$ is a fixed point of g.

(∗) $\overline{\boldsymbol{x}}$ is a fixed point of g if $\overline{\boldsymbol{x}} = g(\overline{\boldsymbol{x}})$.

An Important Note

- Because of the continuity of g, if $\{\boldsymbol{x}_t\}$ converges to a constant limit (denoted by $\tilde{\boldsymbol{x}}$),

$$\tilde{\boldsymbol{x}} = \lim_{t \to \infty} \boldsymbol{x}_{t+1} = \lim_{t \to \infty} f(\boldsymbol{x}_t) = f\left(\lim_{t \to \infty} \boldsymbol{x}_t\right) = f(\tilde{\boldsymbol{x}}), \tag{10}$$

  which yields $\tilde{\boldsymbol{x}} = \overline{\boldsymbol{x}}$.

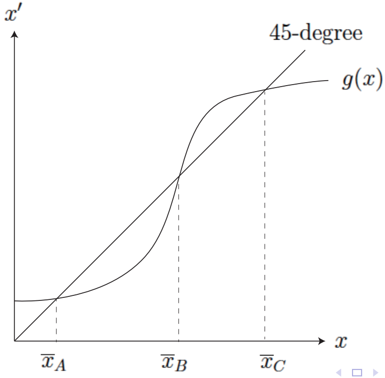
  $\Rightarrow$ we can readily find "If a limit exists, it is a steady state."

- Note that the reverse is NOT necessarily true.

An Example of One Dimensional System ($N = 1$)

- In the figure below, all of $\overline{x}_A$, $\overline{x}_B$ and $\overline{x}_C$ are steady states, but $\{x_t\}$ never converges to $\overline{x}_B$, while it depends on x_0 which state $\{x_t\}$ converges, $\overline{x}_A$ or $\overline{x}_C$.

Figure: An example of multiple steady states



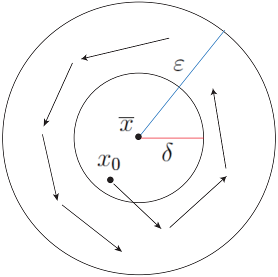
Various Types of Stability

- Lyapunov stability (リアプノフ安定性)

Definition

The steady state $\overline{\boldsymbol{x}}$ is Lyapunov stable, if

$$\forall \varepsilon > 0, \exists \delta \in (0, \varepsilon) \text{ s.t.} \Big(\|\boldsymbol{x}_0 - \overline{\boldsymbol{x}}\| < \delta \Rightarrow \|\boldsymbol{x}_t - \overline{\boldsymbol{x}}\| < \varepsilon \forall t \in \mathbb{N} \Big).$$



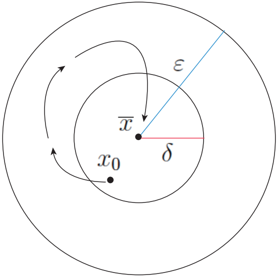
Various Types of Stability

- Asymptotic Stability (漸近安定性)

Definition

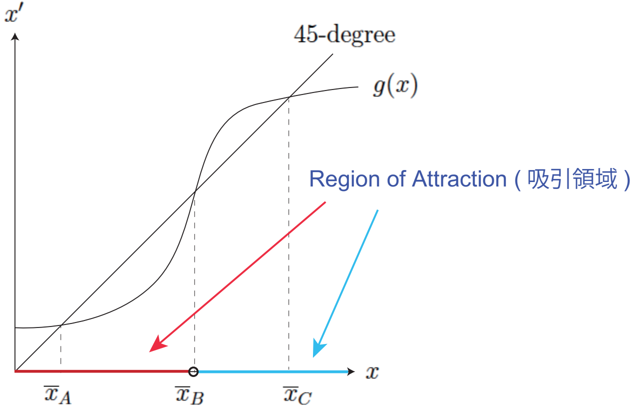
The steady state $\overline{\boldsymbol{x}}$ is asymptotically stable, if it is Lyapunov stable, and δ is chosen such that

$$\|\boldsymbol{x}_0 - \overline{\boldsymbol{x}}\| < \delta \Rightarrow \lim_{t\to\infty} \boldsymbol{x}_t = \overline{\boldsymbol{x}}.$$



Various Types of Stability

- If $\overline{\boldsymbol{x}}$ is not Lyapunov stable, then, it is called unstable.
  $\Rightarrow$ In the above example of one-dimensional dynamics, $\overline{x}_B$ is unstable, whereas both of $\overline{x}_A$ and $\overline{x}_C$ are asymptotically stable.



Various Types of Stability

- If the region of attraction is an entire space of $\boldsymbol{x}$, in other words, if

$$\forall \boldsymbol{x}_0 \in X, \lim_{t \to \infty} \boldsymbol{x_t} = \overline{\boldsymbol{x}},$$

then the steady state $\overline{\boldsymbol{x}}$ is globally asymptotically stable.

- If $\overline{\boldsymbol{x}}$ is Lyapunov stable, but not asymptotically stable, $\{\boldsymbol{x}_t\}$ can converge to a periodic orbit.

Linear Dynamical System

- Then, how do we grasp the stability of the steady state $\overline{\boldsymbol{x}}$?
  $\Downarrow$
- This task is not so difficult, if the system is linear.

  Example: In the Solow growth model presented above, the dynamics $k_{t+1} = sAk_t^{\alpha}$ gives the following solution:

  $$\ln k_t = \ln \overline{k} + \alpha^t (\ln k_t - \ln \overline{k}),$$

  where $\overline{k}$ is the steady state, the value of which is given by $(sA)^{1/(1-\alpha)}$.

  $\Rightarrow$ We can find that $k_t \to \overline{k}$ as $t \to \infty$.

Linear Dynamical System

- Assume that $g(\boldsymbol{x})$ is linear. Specifically, consider the following linear system:

$$\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + B, \tag{11}$$

where A is an $N \times N$ matrix, and B is an $N \times 1$ vector.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}.$$

Assumption

$I - A$ is a non-singler matrix (非特異行列).

$\Downarrow$

- The steady state $\overline{\boldsymbol{x}}$ is given by $(I - A)^{-1}B$.

Linear Dynamical System

- The linear system, (11) is rewritten as

$$\begin{aligned}\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + B &\Leftrightarrow \boldsymbol{x}_{t+1} = A\boldsymbol{x}_t + (I - A)\overline{\boldsymbol{x}} \\ &\Leftrightarrow \boldsymbol{x}_{t+1} - \overline{\boldsymbol{x}} = A(\boldsymbol{x}_t - \overline{\boldsymbol{x}}) \\ &\Leftrightarrow \boldsymbol{z}_{t+1} = A\boldsymbol{z}_t, \text{ where } \boldsymbol{z}_t = \boldsymbol{x}_t - \overline{\boldsymbol{x}}, \qquad (12)\end{aligned}$$

which results in $\boldsymbol{z}_t = A^t\boldsymbol{z}_0$, but this is not very informative.

Linear Dynamical System

- We can solve (12) by taking the following three steps:

Step 1 By solving the following characteristic equation (特性方程式), obtain the eigenvalues λ.

$$\det(A - \lambda I) = 0.$$

- A two-dimensional case:

$$\begin{aligned}\det(A - \lambda I) = 0 &\Leftrightarrow (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0\\ &\Leftrightarrow \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0\\ &\Leftrightarrow \lambda^2 - \mathrm{tr}A + \det A = 0.\end{aligned}$$

Linear Dynamical System

- We can solve (12) by taking the following three steps:

Step 2 Let $\lambda_j (j = 1, 2, \ldots, N)$ denote a eigenvalue, and let

$$\boldsymbol{v}_j = (v_{1j}, v_{2j}, \ldots v_{Nj})^{\mathsf{T}}$$

denote the corresponding eigenvector.
Then,

$$A\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j, \quad j = 1, 2, \ldots, N,$$

or

$$AV = VD,$$

where

$$V = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1N} \\ v_{21} & v_{22} & \cdots & v_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{v_{N1}}_{=\boldsymbol{v}_1} & \underbrace{v_{N2}}_{=\boldsymbol{v}_2} & \cdots & \underbrace{v_{NN}}_{=\boldsymbol{v}_N} \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}.$$

Linear Dynamical System

- We can solve (12) by taking the following three steps:

Step 3 Let $\hat{\boldsymbol{z}}_t = V^{-1}\boldsymbol{z}_t$. Then,

$$\begin{aligned}\hat{\boldsymbol{z}}_{t+1} &= V^{-1}\boldsymbol{z}_{t+1} = V^{-1}A\boldsymbol{z}_t = V^{-1}AV\hat{\boldsymbol{z}}_t \\ &= D\hat{\boldsymbol{z}}_t.\end{aligned}$$

Then, for all $j = 1, 2, \ldots, N$,

$$\hat{z}_{j,t} = c_j(\lambda_j)^t, \tag{13}$$

where c_j is a constant.

Linear Dynamical System

- Once (13) is obtained, we can solve (12) to obtain $\boldsymbol{x}_t$ as follows:

$$\begin{aligned} \boldsymbol{x}_t &= \overline{\boldsymbol{x}} + \boldsymbol{z}_t = \overline{\boldsymbol{x}} + V\hat{\boldsymbol{z}}_t \\ &= \overline{\boldsymbol{x}} + \sum_{j=1}^{N} \boldsymbol{v}_j c_j (\lambda_j)^t, \end{aligned} \tag{14}$$

or equivalently,

$$x_{i,t} = \overline{x}_i + \sum_{j=1}^{N} v_{ij} c_j (\lambda_j)^t \quad i = 1, 2, \ldots, N.$$

Stability of Steady States in Linear Dynamical System

- Now, we are in position to discuss the stability of $\overline{\boldsymbol{x}}$ in the linear system (11).

Theorem

Suppose that in the linear system (11), the matrix A has N distinct real eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_N$. If

$$\mid \lambda_j \mid < 1 \;\forall j = 1, 2, \ldots, N,$$

the steady state $\overline{\boldsymbol{x}}$ is globally asymptotically stable

Proof.

From (14), it is obvious that if $\mid \lambda_j \mid < 1$ for all j, $x_{i,t} \to \overline{x}_i$ as $t \to \infty$ for all i and c_j. $\square$

Stable Manifold

- If $\mid \lambda_j \mid > 1$ for some j, the steady state $\overline{\boldsymbol{x}}$ is not stable.
- If this is the case, it crucially depends on the initial value $\boldsymbol{x}_0$ whether or not $\boldsymbol{x}_t \to \overline{\boldsymbol{x}}$ as $t \to \infty$.

Theorem

Suppose that the matrix A has N distinct real eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_N$, and

1. $\mid \lambda_j \mid < 1$ *for* $j = 1, 2, \ldots, m (< N)$,
2. $\mid \lambda_j \mid > 1$ *for* $j = m+1, m+2, \ldots, N$.

Then, $\boldsymbol{x}_t \to \overline{\boldsymbol{x}}$ if and only if

$$c_j = 0 \text{ for } j = m+1, m+2, \ldots, N. \tag{15}$$

Proof.

Recall (14). □

Stable Manifold

- From (17) (or 17'), it follows that

$$\boldsymbol{x}_0 = \overline{\boldsymbol{x}} + \sum_{j=1}^{N} \boldsymbol{v}_j c_j, \quad \left(\text{or } x_{i,t} = \overline{x}_i + \sum_{j=1}^{N} v_{ij} c_j \quad i = 1, 2, \ldots, N \right). \tag{16}$$

Then, the above theorem characterizes the condition for the $\boldsymbol{x}_0$ under which $\boldsymbol{x}_t \to \overline{\boldsymbol{x}}$.

- Define the set Ψ by

$$\Psi = \{\boldsymbol{x}_0 \in X \mid c_j = 0 \text{ for } j = m+1, m+2, \ldots, N\}. \tag{17}$$

  - Briefly speaking, Ψ is the set of $\boldsymbol{x}_0$, where $\boldsymbol{x}_t \to \overline{\boldsymbol{x}}$ as $t \to \infty$ for all $\boldsymbol{x}_0 \in \Psi$.
  - Ψ, which is a subspace of $\mathbb{R}^m$, is called the stable manifold (安定多様体),

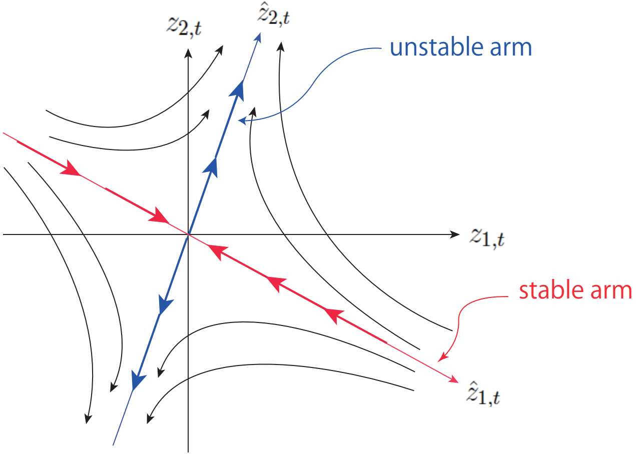
Two-dimensional Case

- Consider the linear system of $N = 2$.

1. $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid < 1$ $(m = 2)$ $\Rightarrow$ $\overline{\boldsymbol{x}}$ is globally asymptotically stable.
   $\Rightarrow$ $\overline{\boldsymbol{x}}$ is called a sink (沈点).
2. $\mid \lambda_1 \mid > 1$ and $\mid \lambda_2 \mid > 1$ $(m = 0)$ $\Rightarrow$ $\overline{\boldsymbol{x}}$ is called a source (源点).
3. $\mid \lambda_1 \mid < 1$ and $\mid \lambda_2 \mid > 1$ $(m = 1)$ $\Rightarrow$ $\overline{\boldsymbol{x}}$ is called saddle (鞍点).

Saddle Point Stability

- In the case of $\overline{\boldsymbol{x}}$ being a saddle, the dynamics of $\boldsymbol{x}_t - \overline{\boldsymbol{x}}(= \boldsymbol{z}_t)$ is depicted as



$(*)$ The set of the $\boldsymbol{z}_0(= \boldsymbol{x}_0 - \overline{\boldsymbol{x}})$ *on the stable arm* corresponds to the stable manifold.

Saddle Point Stability

- If $\boldsymbol{z}_0 (= \boldsymbol{x}_0 - \overline{\boldsymbol{x}})$ is on the stable arm, the economists say that the steady state $\overline{\boldsymbol{x}}$ is saddle point stable (鞍点安定).

  In the Ramsey-Cass-Koopmans model, economic implications of the saddle-point stability will be discussed.

Nonlinear System

- Now we consider the dynamical system is given by the nonlinear difference equations.

$$\boldsymbol{x}_{t+1} = g(\boldsymbol{x}_t). \tag{18}$$

1. Lyapunov's direct method (omitted)
2. Local stability

Linearized System of Nonlinear Dynamics

Assumption

$g : X \to X$ is C^1 class.

- Linearization of $g(\boldsymbol{x})$ around the steady state $\overline{\boldsymbol{x}}$:

$$\boldsymbol{x}_{t+1} \simeq \overline{\boldsymbol{x}} + Dg(\overline{\boldsymbol{x}})(\boldsymbol{x}_t - \overline{\boldsymbol{x}}), \tag{19}$$

where Dg is the Jacobian matrix (ヤコビ行列)

Theorem (Local Stability)

Suppose that in the nonlinear system (18), the Jacobian matrix Dg of its linearized system (19) has N distinct real eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_N$. If

$$\mid \lambda_j \mid < 1 \quad \forall j = 1, 2, \ldots, N,$$

the steady state $\overline{\boldsymbol{x}}$ is locally asymptotically stable.