Advanced Macroeconomics

(Department of Social Engineering, Spring FY2015)

Dynamic Optimization in Discrete Time (2): Infinite-horizon Dynamic Programming

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Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures incl. today)
  2. The Ramsey-Cass-Koopmans Model (3 lectures)
  3. Endogenous Growth Models (2 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic Dynamic Programming (3 lectures)

Plan of Lecutres in the Part of “Dynamic Optimziation”

- April, 8 (Wed): An introduction to dynamic optimization
- April, 15 (Wed) (Today): Infinite-horizon dynamic programming
- April, 22 (Wed): System of difference equations and its stability
- April, 30 (Thu): Continuous-time optimal control
  (∗) Note the day of week, not the same as usual.

Corrigendum

- On p. 20,
  - for "$W_t < W$" read "$W_t \leq W$";
- On p.23,
  - In (9), for "$\lambda_T W_{T+1} = 0$" read "$\beta^{T-1}\lambda_T W_{T+1} = 0$"
  - for "$\lambda_T W_{T+1}$ is called..." read "$\beta^{T-1}\lambda_T W_{T+1} = 0$ is called ..."

Introduction

- So far, we have considered a many, but finite-period case.

  $\Rightarrow$ The consumption streams over T periods is denoted by $\boldsymbol{c} = (c_1, c_2, \ldots, c_T)$.

- However, thinking of T being infinite is a good "approximation," when we consider open-ended situations.

- Hereafter, we assume that time extends to 0 to infinity.

  $\Rightarrow$ Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers;

Notations

Let

- $c_t \in C \subseteq \mathbb{R}_+$ be the control variable (制御変数);
- $x_t \in X \subseteq \mathbb{R}_+$ be the state variable (状態変数);

$\Rightarrow \{c_t\}_{t=0}^{\infty}$ and $\{x_t\}_{t=0}^{\infty}$ be their sequences;

$(*)$ Assumption that both of them are scalars is made only for simplicity. Needless to say, each of them can be a vector.

Notation (cont'd)

- $F : X \times C \to \mathbb{R}$ be the one-period return function (一期収益関数); and
- $G : X \times C \to X$ be the transition function (推移関数);

  $\Rightarrow$ This gives the transition equation: $x_{t+1} = G(x_t, c_t)$.
- $\beta \in (0, 1)$: the discount factor (割引因子)

Infinite-horizon Optimization Problem

- The infinite-horizon discounted optimization problem is generally given by

$$\begin{aligned} \max_{\{c_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t F(x_t, c_t) \\ \text{s.t.} \quad & x_{t+1} = G(x_t, c_t), \quad (x_t, c_t) \in X \times C, \quad t = 0, 1, 2, \ldots. \\ & x_0 \in X \text{ given} \end{aligned} \tag{P0}$$

- If you specify $F(x, c) = u(c)$ and $G(x, c) = x - c$, you can immediately recover the infinite-horizon counterpart of cake-eating problem.

Cake-eating Example Reconsidered

- Go back to the cake-eating problem.
- Substituting the transition equation, $c_t = W_t - W_{t+1}$, into $u(c_t)$, we can define the following function v:

$$v(W_t, W_{t+1}) \equiv u(W_t - W_{t+1})$$

- Then, the cake-eating problem is expressed more simply:

$$\begin{aligned} \max_{\{W_t\}_{t=1}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t v(W_t, W_{t+1}) \\ \text{s.t.} \quad & W_{t+1} \in [0, W_t] \\ & W_0 = W \end{aligned}$$

Infinite-horizon Optimization Problem

- Hereafter, we assume that the problem (P0) can be expressed as the following reduced form:

$$\begin{aligned} \max_{\{x_t\}_{t=1}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in \Gamma(x_t) \\ & x_0 \in X \text{ given} \end{aligned} \tag{P}$$

  where

  - $f : X \times X \to \mathbb{R}$ is a reduced form of the one-period return function, generated from F and G;
  - $\Gamma : X \to X$ is the correspondence, whose graph is

$$\{(x, y) \in X \times X \mid y \in \Gamma(x)\}.$$

Preliminary

- Given $x_t \in X$, a choice $\tilde{x}_{t+1}$ is feasible if $\tilde{x}_{t+1} \in \Gamma(x_t)$.
- Given $x_0 \in X$, let

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \mid x_{t+1} \in \Gamma(x_t) \forall t \in \mathbb{Z}_+\}$$

Definition

Given $x_0 \in X$, any sequence $\{\tilde{x}_t\}_{t=1}^{\infty} \in \Pi(x_0)$ is called the feasible path (or plan) (実行可能経路).

Assumptions

Assumption

For all $x \in X$, $\Gamma(x)$ *is nonempty.*

Assumption

For all $x_0 \in X$ *and* $\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t f(x_t, x_{t+1})$ *exists in* $\mathbb{R}$.

($*$) In Stokey and Lucas (1989, Ch. 4), the second assumption is relaxed so that the finite sum $\sum_{t=0}^{n} \beta^t f(x_t, x_{t+1})$ can diverge: i.e., they assume $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t f(x_t, x_{t+1})$ exists in $\mathbb{R} \cup \{+\infty, -\infty\}$.

Value Function

Definition (Value function)

The value function $V^* : X \to \mathbb{R}$ is defined as

$$V^*(x_0) = \max_{\{x_t\}_{t=1}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}) \;\middle|\; x_{t+1} \in \Gamma(x_t) \forall t \in \mathbb{Z}_+ \right\}, \qquad (1)$$

or more simply

$$V^*(x_0) = \max_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}). \qquad (1')$$

Bellman Equation

Definition (Bellman equation)

The following functional equation is called the Bellman equation (ベルマン方程式)

$$V(x) = \max_{x' \in \Gamma(x)} \left\{ f(x, x') + \beta V(x') \right\}. \tag{2}$$

Principle of Optimality: Necessity

Theorem

The value function V^* defined in (1) satisfies the Bellman equation (2).

Proof.

Proof will be given in the supplementary materials. □

Principle of Optimality: Sufficiency

Theorem

Given $x_0 \in X$, *let* $\{x_t^*\}_{t=1}^{\infty}$ *denote the sequence generated by solving the Bellman equation (2). Suppose that* $\{x_t^*\}_{t=1}^{\infty} \in \Pi(x_0)$ *and the following boundary condition (境界条件) is satisfied:*

$$\lim_{t\to\infty} \beta^t V(x_t^*) = 0.$$

Then, $\{x_t^*\}_{t=0}^{\infty}$ *is the solution to the problem (1).*

Proof.

Proof will be given in the supplementary materials. □

Policy Function

Definition (Policy function)

$h : X \to X$ is called the policy function (政策関数) if

$$h(x) = \arg \max_{x' \in \Gamma(x)} \{f(x, x') + \beta V(x')\} \tag{3}$$

How to Obtain V^*

- In summary,
  - From the first theorem, V^* satisfies the Bellman equation (2);
  - Note that (2) may have other solutions. However, the second theorem shows that as long as the boundary condition is satisfied, a solution to (2) is V^*.

$\Rightarrow$ We can focus on (2) instead of the original problem (P).

- Furthermore, if we can obtain the value function V^* from the Bellman equation, we can express the sequence $\{x_t^*\}_{t=1}^{\infty}$ in the following recursive form:

$$\forall x_0 \in X, \quad x_{t+1}^* = h(x_t^*), \quad t = 0, 1, 2, \ldots \tag{4}$$

How to Obtain V^*

- Conversely, we *have to* obtain the value function from the Bellman equation.
- How?
  1. Guess and verify (推測と確認)
  2. Value function iteration (価値観数の繰り返し計算)

     $\ddots$

Guess and Verify

- If we specify the functional form, we can obtain V^* by the method of guess and verify.
- If we specify $u(c) = \ln c$ in the infinite-horizon cake-eating problem on pp. 9, we get

$$V^*(W) = \frac{1}{1-\beta} \ln W + (1-\beta)^{-1} \left[\ln(1-\beta) + \frac{\beta}{1-\beta} \ln \beta \right].$$

We also obtain the policy function as $W' = \beta W$.

Value Function Iteration

- Given any V, define T by

$$T(V)(x) = \max_{x' \in \Gamma(x)} \{f(x, x') + \beta V(x')\}. \tag{5}$$

  T is called the Bellman operator.

  - $T : C(X) \to C(X)$, where $C(X)$ is a space of continuous function on X.

- At first, *arbitrarily* choose a function, say, $V_0(x) \in C(X)$, and substitute this into the right-hand-side of (5) for V.
  $\Downarrow$
- Then, in (5), the operator T gives the new function, say, $V_1(x)$.
  $\Downarrow$
- Substitute V_1 into the RHS of (5) for V.

  $\ddots$

Value Function Iteration

- Briefly speaking, the functional sequence, $\{V_j(x)\}_{j=0}^{\infty}$ is generated by the Bellman operator.

- Therefore, if $V_j(x)$ uniformly converges to $V^*(x)$, we can obtain the value function.

  ($*$) In Theorem 4.6 of Stokey and Lucas (1989, Ch. 4), it is shown that the operator $T : C(X) \to C(X)$ is a contraction mapping, which in turn shows that

$$T(V^*) = V^*, \quad \lim_{j\to\infty} T^j(V_0) = V^* \forall V_0 \in C(X).$$

  (Proof is omitted here) Then, V_j uniformly converges to V^*.

Euler Equation

- If V is differentiable, the problem given by the Bellman equation (2) has the following first-order-condition:

$$f_2(x, x') + \beta V_x(x') = 0,$$

where $f_2 = \partial f(x, x')/\partial x'$.

- Given V, the above condition gives the policy function implicitly, $x' = h(x)$.

- Substituting back into the Bellman equation, we have

$$V(x) = f(x, h(x)) + \beta V(h(x)). \tag{6}$$

Euler Equation

- Differentiating (6) with respect to x yields:

$$\begin{aligned} V_x(x) &= f_1(x,x') + \underbrace{\left(f_2(x,x') + \beta V_x(x')\right)}_{=0 \text{ (envelop theorem)}} h'(x) \\ &= f_1(x,x') \end{aligned}$$

- Then, F.O.C is rewritten as

$$f_2(x,x') + \beta f_1(x',x'') = 0,$$

or, if we use the time script,

$$f_2(x_{t-1},x_t) + \beta f_1(x_t,x_{t+1}) = 0. \qquad (7)$$

(7) is called the Euler equation (オイラー方程式).

Transversality Condition

- In the Euler equation (7), the unknown function V disappears.
- Therefore, the Euler equation is very useful, when we face the difficulty of finding V^* directly, but can verify V^* is differentiable.
- However, the Euler equation is *necessary* for maximization, but not sufficient.
  $\Downarrow$
  The next theorem gives the sufficient conditions for the problem (P).

Transversality Condition

Theorem

Suppose that $f(x, x')$ is increasing in x, concave and continuously differentiable in (x, x'). Then, given $x_0 \in X$, the sequence $\{x_t^*\}_{t=1}^{\infty} \in \Pi(x_0)$ is the solution to (P) if it satisfies the Euler equation (7), and

$$\lim_{t \to \infty} \beta^t f_1(x_t^*, x_{t+1}^*) x_t^* = 0. \tag{8}$$

Proof.

Proof will be given in the supplementary materials. □

(8) is the transversality condition (横断性条件) in an infinite-horizon problem.