Advanced Macroeconomics

(Department of Social Engineering, Spring FY2015)

Dynamic Optimization in Discrete Time (1)

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Purpose of This Course

- **Purpose:** This course is aimed at providing students with standard methods in modern dynamic macroeconomics.
  - What is modern macroeconomics?
  - Shortly speaking, it extends the Solow model by applying the dynamic optimization techniques to endogenize households' saving rate.
    So, a knowledge of the Solow model is a prerequisite for this course.
- **Office Hour:** Room 636, Wednesdays 10:45–12:15.

  (∗) If you want to meet with me at other times, appointment is required.
- **E-mail address:** ohdoi@soc.titech.ac.jp

Grading Policy

- **Grading Policy:** The final grade is based on homework assignments (30%), midterm- (30%) and final examinations (40%).
- The midterm and final exams are tentatively scheduled on
    - June 3 (Midterm)
    - July 29, or August 5 (yet to be finalized) (Final)

Homework Assignments

- I will give you homework assignments X times, where the number of X is TBA (4-5 times).

- If students submit an assignment by the due date, I will give them at most $30/X$ points. Therefore, if they submit all assignments with correct answers, you will receive 30 points.

  $(*)$ You can hand-in the assignments in the language whichever you prefer, English or Japanese.

Course Guideline

- **Course Guideline**
  1. Dynamic Optimization (4 lectures incl. today)
  2. The Ramsey-Cass-Koopmans Model (3 lectures)
  3. Endogenous Growth Models (2 lectures)
  4. Models of Time-inconsistent Preferences (Preference Reversals) (1-2 lectures)
  5. Some Macroeconomic Applications of Stochastic Dynamic Programming (3 lectures)

Plan of Lecutres in the Part of "Dynamic Optimziation"

- April, 8 (Wed) (Today): An introduction to dynamic optimization
- April, 15 (Wed): Infinite-horizon dynamic programming
- April, 22 (Wed): System of difference equations and its stability
- April, 30 (Thu): Continuous-time optimal control
  (∗) Note the day of week, not the same as usual.

Introduction: A Cake-eating Problem

- Suppose that you are present with a cake of size $W > 0$.
- In each period $t \ (= 1, 2, 3, \ldots, T)$, you can eat some of the cake, and save the rest.
- Let
  - $c_t \geq 0$ be amount of your consumption in period t;
  - $u : \mathbb{R}_+ \to \mathbb{R}$ be your one-period utility function from c_t;
  - $\boldsymbol{c} = (c_1, c_2, \ldots, c_T) \in \mathbb{R}_+^T$.

Your Preferences

- Your preferences are assumed to be given by the function $U : \mathbb{R}_+^T \to \mathbb{R}$:

$$
\begin{aligned}
U(\boldsymbol{c}) &= u(c_1) + \beta u(c_2) + \ldots + \beta^{T-1} u(c_T) \\
&= \sum_{t=1}^{T} \beta^{t-1} u(c_t),
\end{aligned}
$$

where $\beta \in (0, 1)$ is called the discount factor (割引因子).

- If one defines $\rho > 0$ such that

$$
\beta \equiv \frac{1}{1+\rho}.
$$

ρ is called the discount rate (割引率).

Optimization Problem

- Question: How do you decide your optimal plans of eating the cake?
- From the view point of microeconomic theory, the problem is formulated as the utility maximization problem as follows:

$$\begin{aligned} \max_{\boldsymbol{c}} \quad & \sum_{t=1}^{T} \beta^{t-1} u(c_t), \\ \text{s.t.} \quad & \sum_{t=1}^{T} c_t \leq W, \\ & c_t \geq 0 \quad t = 1, 2, \ldots, T. \end{aligned}$$

$(*)$ From the above constraints, $c_t \leq W$ automatically implies for all t.

Optimization Problem

- Thus, in this simple example, there is no
  - Trade in markets: you do not buy or sell the cake;
  - Strategic interactions between you and other people: you do not need to share the cake with any others.
- Instead, this example focus on your own *intertemporal choice* of consumption, which gives the benchmark for the analysis of an individual's saving-consumption decision in macroeconomics.

Optimization Problem

- Let $\mathcal{D}$ denote the constraint set (制約集合):

$$\mathcal{D} = \left\{ \boldsymbol{c} \in \mathbb{R}_+^T \middle| \sum_{t=1}^{T} c_t \leq W \right\}$$

$\Rightarrow \mathcal{D}$ is compact.

Assumption

$U : \mathbb{R}_+^T \to \mathbb{R}$ *is a continuous function on* $\mathcal{D}$.

$\Downarrow$

- The Weierstrass Theorem:
  U attains a maximum (and a minimum) on $\mathcal{D}$.

Optimization Problem

- Exercise: Suppose that $u(c_t) = c_t$ for all $t = 1, 2, \ldots, T$. Then, show that U is maximized at

$$c_1 = W, \quad c_2 = c_3 = \ldots = c_T = 0.$$

Inequality-constrained Optimization

- The above problem is the inequality-constrained optimization problem.
- Hereafter, in addition to its continuity, we assume that $u(c)$ is
  1. continuously differentiable (a necessary number of times);
  2. strictly increasing ($u'(c) > 0$);
  3. strictly concave ($u''(c) < 0$) $\Rightarrow U(\boldsymbol{c})$ is strictly concave.
- Furthermore, we assume

$$\lim_{c \to 0} u'(c) = +\infty,$$

which is called the Inada condition (稲田条件).

Using the Theorem of Kuhn and Tucker: A "Cookbook" Procedure

- Construct the following Lagrangian:

$$L(\boldsymbol{c}, \lambda, \boldsymbol{\mu}) = U(\boldsymbol{c}) + \lambda \left(W - \sum_{t=1}^{T} c_t \right) + \sum_{t=1}^{T} \mu_t c_t,$$

where

- λ: The KT multiplier associated with the constraint: $\sum_{t=1}^{T} c_t \leq W$;
- μ_t: That associated with the constraint $c_t \geq 0$, and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_T)$.

Using the Theorem of Kuhn and Tucker: A "Cookbook" Procedure

- Then, derive the first order conditions (F.O.Cs):

$$\beta^{t-1}u'(c_t) + \mu_t = \lambda, \quad t = 1, 2, \ldots, T, \tag{1}$$

$$\sum_{t=1}^{T} c_t \leq W, \ \lambda \geq 0, \ \lambda \left(W - \sum_{t=1}^{T} c_t \right) = 0, \tag{2}$$

$$c_t \geq 0, \ \mu_t \geq 0, \mu_t c_t = 0. \tag{3}$$

- (2) and (3) are called the complementary slackness condition (相補性条件).

- Thanks to the concavity of U and the fact that $\mathcal{D}$ is a convex set, the above F.O.Cs provide necessary and sufficient conditions for the maximization problem.

Solution

- Let $\boldsymbol{c}^*$ denote the solution of the problem. Thanks to $u'(c) > 0$ for all c and the Inada condition $\lim_{c\to 0} u'(c) = +\infty$, From (2) and (3), we have

$$\sum_{t=1}^{T} c_t^* = W, \tag{4}$$

$$c_t^* > 0 \; t = 1, 2, \ldots, T. \tag{5}$$

Solution

- From $c_t^* > 0$ and (3), $\mu_t = 0$.
- Substituting this result into (1), we have

$$\beta^{t-1} u'(c_t^*) = \lambda, t = 1, 2, \ldots, T.$$

$\Rightarrow$ c_t is obtained as $c_t^* = (u')^{-1}(\lambda/\beta^{t-1})$, where $(u')^{-1}$ is the inverse function of u.

$\Rightarrow$ Substituting this result into (4),

$$\sum_{t=1}^{T} \beta^{t-1} \underbrace{(u')^{-1}(\lambda/\beta^{t-1})}_{c_t^*} = W.$$

Thus, by specifying the functional form of u, we can solve the above equation for λ, which in turn determines the value of c_t^*.

Euler Equation

- Hereafter, we let the solution $\boldsymbol{c}^* = (c_1^*, c_2^*, \ldots)$ denote the optimal consumption plan (最適消費計画).

- At the same time, we can readily obtain the following relationship:

$$\begin{aligned} &\beta^{t-1}u'(c_t^*) = \lambda, \quad t = 1, 2, \ldots, T, \\ &\Rightarrow u'(c_t^*) = \beta u'(c_{t+1}^*), \quad t = 1, 2, \ldots, \end{aligned} \tag{6}$$

  (6) is called the Euler equation (オイラー方程式).

- Economic meanings of the Euler equation will be discussed in the Ramsey-Cass-Koopmans model.

Reformulation of the Cake-eating Problem

- Let W_t denote the size of the leftover cake, which remains to be available for you in period t;

$$\begin{aligned} W_1 &= W, \\ W_t &< W \quad t = 2, 3, \ldots, T+1. \end{aligned}$$

- Then, the value of W_t changes over time according to the following law of motion:

$$W_{t+1} = W_t - c_t. \tag{7}$$

(7) is called the transition equation (推移方程式).

Reformulation of the Cake-eating Problem

- Then, the cake-eating problem can be formulated also as

$$\begin{aligned} \max \quad & U(\boldsymbol{c}) = \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\ \text{s.t.} \quad & W_{t+1} = W_t - c_t, \quad t = 1, 2, \dots, T \\ & W_{T+1} \geq 0, \; W_1 = W. \end{aligned}$$

  $(*)$ The inequality constraint, $c_t \geq 0$, is now omitted because we have already known that it never binds owing to the Inada condition.

- W_{T+1} is amount of a leftover piece of cake in period T.
  $\Rightarrow \sum_{t=1}^{T} c_t = W$ if and only if $W_{T+1} = 0$.

  The above problem is called the optimal control problem (最適制御問題) in discrete time.

Optimal Control and Transversality Condition

- Once reset the meanings of notations, λ and μ, defined above.
- Construct the following Lagrangian :

$$
\begin{aligned}
L &= \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \sum_{t=1}^{T} \tilde{\lambda}_t (W_t - c_t - W_{t+1}) + \mu W_{T+1} \\
&= \sum_{t=1}^{T} \beta^{t-1} \left[u(c_t) + \lambda_t (W_t - c_t - W_{t+1}) \right] + \mu W_{T+1}
\end{aligned}
$$

where

- $\lambda_t (= \beta^{-(t-1)} \tilde{\lambda}_t)$: The multiplier associated with the transition equation;
- μ: The KT multiplier associated with the constraint $W_{t+1} \geq 0$.

$(*)$ λ_t is called the costate variable (共役変数) in the context of control.

Optimal Control and Transversality Condition

- Deriving the F.O.Cs, and arranging them,

$$u'(c_t) = \beta u'(c_{t+1}), \tag{8}$$

$$W_{T+1} \geq 0, \ \lambda_T \geq 0, \ \lambda_T W_{T+1} = 0. \tag{9}$$

- (8) is the Euler equation, while in (9), $\lambda_T W_{T+1}$ is called the transversality condition (横断性条件), TVC.

- The role of the Euler equation and the TVC will be examined again in the part of continuous-tine optimal control.

Recursive Feature of the Problem

- So far, we formulate the cake-eating problem in two different ways.
- Note that, in either case, you solved the problem *in the initial period*.
- Then, suppose that you stop and reconsider the problem in period, say, t_0. Then, your problem from then on is

$$\begin{aligned} \max \quad & \sum_{t=t_0}^{T} \beta^{t-t_0} u(c_t) \\ \text{s.t.} \quad & W_{t+1} = W_t - c_t, \\ & W_{T+1} \geq 0, \quad W_{t_0} \text{ given.} \end{aligned}$$

$\Rightarrow$ You will solve essentially the same problem as you did in the initial period.

Recursive Feature of the Problem

- The dynamic programming technique, based on the Bellman's principle of optimality, utilizes such a property that the problem is recursively defined.
- Let

$$V(W_1) = \max_{\boldsymbol{c}} \left\{ \sum_{t=1}^{T} \beta^{t-1} u(c_t) : W_{t+1} = W_t - c_t, t = 1, 2, \dots T \right\},$$

where $V : \mathbb{R}_+ \to \mathbb{R}$ is called the value function (価値関数). In the context of economics, V is called the indirect utility function.

Bellman Equation

- Briefly speaking, the principle of optimality means

$$
\begin{aligned}
V(W_1) &= \max_{\boldsymbol{c}} \left\{ \sum_{t-1}^{T} \beta^{t-1} u(c_t) : W_{t+1} = W_t - c_t, t = 1, 2, \dots T \right\} \\
&= \max_{c_1} \left\{ u(c_1) + \beta V(W_2) : W_2 = W_1 - c_1 \right\}. \qquad (10)
\end{aligned}
$$

  (10) is called the Bellman equation (ベルマン方程式).

- The dynamic programming technique, now widely used in macroeconomics, solves the maximization problem by converting the original problem into a two-period problem characterized in the Bellman equation.

  (∗) Note that the value function in (10) is *still to be determined.* Thus, the Bellman equation is a functional equation.

Next Week

- Infinite-horizon dynamic programming with more general functional forms