# Fundamentals of Mathematical and Computing Sciences: Applied Mathematical Sciences

## PART II: A Second Course in Probability

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Room W832
Evaluation by Reports

## Outline of the Lecture

This course introduces several basic concepts of mathematical optimization, probability and statistics, and is intended to provide key knowledge necessary for advanced study in Mathematical and Computing Sciences.

## References

[1] S. M. Ross and E. A. Peköz (2007). *A Second Course in Probability.* www.ProbabilityBookstore.com, Boston.

[2] A. Gut (2013). *Probability: A Graduate Course, Second Edition.* Springer, New York.

# 1 Probability Space Revisited

**Def. 1.1**

**Probability Space** $(\Omega, \mathcal{F}, \mathsf{P})$

$\Omega$**: Sample space**
Set of all possible outcomes (of a probabilistic phenomenon)

$\mathcal{F}$**:** $\sigma$**-Field** (or $\sigma$-algebra) on $\Omega$
Set of subsets of $\Omega$ on which probability is defined (detailed later)

$(\Omega, \mathcal{F})$**: Measurable space**

**Event:** An element of $\mathcal{F}$

$\mathsf{P}$**: Probability measure** on $(\Omega, \mathcal{F})$
Set function from $\mathcal{F}$ to $[0, 1]$ (detailed later)

$\mathsf{P}(A)$, $A \in \mathcal{F}$**:** Probability of event $A$

**Questions:**

- Why is probability $\mathsf{P}$ set function?
  (Can not we assign the probability to each element of $\Omega$?)
- What is $\sigma$-field? Why is it necessary?

## 1.1 Discrete Probability Space

When $\Omega$ is a countable set; $\Omega = \{\omega_1, \omega_2, \ldots\}$, we can assign a probability value to each element of $\Omega$.

**Def. 1.2 (Discrete Probability Space)**

**Probability (mass) function** $p$**:** $\Omega \to [0, 1]$ s.t. $\displaystyle\sum_{\omega \in \Omega} p(\omega) = 1$

$\mathcal{F} = 2^\Omega$**:** Power set (set of all subsets) of $\Omega$

$$\mathsf{P}(A) = \sum_{\omega \in A} p(\omega), \quad A \in \mathcal{F}$$

**Example 1.1 (Coin tosses)** $\Omega = \{\omega_1, \omega_2, \ldots\}$

$\omega_i$: 1st head appears at the $i$th toss.

$$p(\omega_i) = \left(\frac{1}{2}\right)^i, \quad i = 1, 2, \ldots, \quad \Rightarrow \quad \sum_{i=1}^{\infty} p(\omega_i) = \frac{1/2}{1 - 1/2} = 1$$

$A = \{\omega_1, \omega_2, \ldots, \omega_n\} \Rightarrow$ Probability that 1st head appears in 1st $n$ tosses.
$A = \{\omega_2, \omega_4, \omega_6, \ldots\} \Rightarrow$ Probability that 1st head appears in even tosses.

## 1.2 $\sigma$-Fields

If we want to assign probability values to elements of an uncountable sample space, e.g., $\Omega = [0, 1]$, we can only assign positive values to at most countable number of elements.
$\Rightarrow$ Probability is defined by assigning values to subsets of $\Omega$.

**Example 1.2** $\Omega = [0, 1]$, $\mathsf{P}([a, b]) = b - a$ for $0 \leq a < b \leq 1$.

**Question:** On which set $\mathcal{F}$ of subsets, is probability $\mathsf{P}$ well defined?
(What is the domain of set function $\mathsf{P}$?)

**Requirements:**

- $\Omega \in \mathcal{F}$ (Probability is assigned to $\Omega$ itself)
- $\mathcal{F}$ is closed w.r.t. set operations ($^c$, $\cup$, $\cap$)
  $A$, $B \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, $A \cup B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$

$\sigma$-Fields satisfy these requirements.

**Def. 1.3 ($\sigma$-Field or $\sigma$-Algebra)**

Set of subsets $\mathcal{F}$ of $\Omega$ is a **$\sigma$-field** (or **$\sigma$-algebra**) on $\Omega \Leftrightarrow$

1. ${}^\exists A \subset \Omega$ s.t. $A \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$(\Omega, \mathcal{F})$**: Measurable space**

**Properties of $\sigma$-fields I**

i) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$

ii) $A$, $B \in \mathcal{F} \Rightarrow A \cup B$, $A \cap B$, $A \setminus B$ $(= A \cap B^c) \in \mathcal{F}$

iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

A $\sigma$-field is closed w.r.t. infinite set operations.

**Question**
Why should the domain of $\mathsf{P}$ be closed w.r.t. infinite set operations?

**Example 1.3** $X\colon \Omega \to \mathbb{R}$ is a random variable $\Leftrightarrow$

$$^{\forall}a \in \mathbb{R}, \quad \{X \leq a\} = \{\omega \in \Omega : \ X(\omega) \leq a\} \in \mathcal{F} \quad \text{(detailed later)}$$

$X$, $Y$: Random variables $\Rightarrow X + Y$: Random variable?

$$\{X + Y > a\} = \bigcup_{n=1}^{+\infty} \bigcup_{m=-\infty}^{+\infty} \left\{X > \frac{m}{2^n}\right\} \cap \left\{Y > a - \frac{m}{2^n}\right\} \in \mathcal{F}$$

If $\mathcal{F}$ would not be closed w.r.t. infinite set operations, $X + Y$ could not be a random variable.

**Properties of $\sigma$-fields II**

iv) $\sigma$-fields are not unique for a sample space $\Omega$
Examples:

- $\mathcal{F}_0 = \{\emptyset, \Omega\}$
- $\mathcal{F}_A = \{\emptyset, A, A^c, \Omega\}$ for a nonempty $A \subset \Omega$
- $\mathcal{F}_* = 2^{\Omega} = \{\text{all possible subsets of } \Omega\}$ (power set)

v) For $\sigma$-fields $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_1 \cup \mathcal{F}_2$ may not be a $\sigma$-field
$A$, $B \subset \Omega$ $(A \neq B)$, $A \cup B \notin \mathcal{F}_A \cup \mathcal{F}_B$

**Lem. 1.1 (Uncountable intersection of $\sigma$-fields)**

$\mathcal{X}$**:** An uncountable set

$\mathcal{F}_x$**,** $x \in \mathcal{X}$**:** A collection of $\sigma$-fields

$\Rightarrow \bigcap_{x \in \mathcal{X}} \mathcal{F}_x$ is a $\sigma$-field

**Lem. 1.2 ($\sigma$-Field generated by a given set of subsets)**

$\mathcal{A}$**:** Set of subsets of $\Omega$

$\Rightarrow$ The smallest $\sigma$-field containing $\mathcal{A}$ (intersection of all $\sigma$-fields containing $\mathcal{A}$), denoted by $\sigma(\mathcal{A})$, can be constructed.

**Example 1.4** $\Omega = \{a, b, c\}$. $\mathcal{A} = \{\{a, b\}, \{c\}\}$ is not a $\sigma$-field
$\Rightarrow \sigma(\mathcal{A}) = \{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$.

## 1.3 Borel Fields

$\sigma$-Fields are not unique for a given $\Omega \Rightarrow$ Which $\sigma$-field should be chosen?

- $\Omega$ is countable $\Rightarrow \mathcal{F} = 2^\Omega$ (set of all subsets of $\Omega$) is sufficient.
- $\Omega$ is uncountable $\Rightarrow \mathcal{F} = 2^\Omega$ is not good!

**Example 1.5 (Vitali sets)** $\Omega = [0, 1]$, $\alpha$: an irrational number

1. Define an equivalence relation "$\sim$" by

$$x \sim y, \quad x, y \in [0, 1] \quad \Leftrightarrow \quad {}^{\exists}n \in \mathbb{Z} \text{ s.t. } y = x + n\,\alpha \pmod 1$$

2. Divide $\Omega = [0, 1]$ to (continuously infinite) equivalent classes by "$\sim$" (each class is a countable set)
3. Make a sequence of sets $\ldots, A_{-1}, A_0, A_1, A_2, \ldots$ by
   - $A_0 = \{$one representative from each equivalent class$\}$
   - $A_n = \{x + n\,\alpha \pmod 1 \mid x \in A_0\}, \quad n = \pm 1, \pm 2, \ldots$

   (Each $A_n$, $n \in \mathbb{Z}$, is an uncountable set)

It is not possible to define the uniform distribution on $(\Omega, 2^\Omega)$
(which should satisfy $\mathsf{P}(A_n) =$ constant, $n \in \mathbb{Z}$, and $\sum_{n=-\infty}^{\infty} \mathsf{P}(A_n) = 1$).

**Def. 1.4 (Borel Field)**

$(E, d)$**:** Metric space

$\mathcal{E}$**:** Set of all open subsets in $E$

**Borel field $\mathcal{B}(E)$ on $E$:** $\sigma$-field $\sigma(\mathcal{E})$ generated by $\mathcal{E}$
(smallest $\sigma$-field containing $\mathcal{E}$)

**Example 1.6 (Borel field on $\mathbb{R}$)**
$\mathcal{B}(\mathbb{R})$: $\sigma$-field generated by the set of all open intervals in $\mathbb{R}$

- ${}^{\forall}a, b \in \mathbb{R}$ $(a < b)$, $(a, b) \in \mathcal{B}(\mathbb{R})$

- $\{a\} = \bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, a + \frac{1}{n}\right) \in \mathcal{B}(\mathbb{R})$,
  $[a, b) = \{a\} \cup (a, b) \in \mathcal{B}(\mathbb{R})$, $(a, b] = (a, b) \cup \{b\} \in \mathcal{B}(\mathbb{R})$,
  $[a, b] = [a, b) \cup \{b\} \in \mathcal{B}(\mathbb{R})$

- $(a, \infty) = \bigcup_{n=1}^{\infty}(a, a + n) \in \mathcal{B}(\mathbb{R})$,
  $(-\infty, a] = (a, +\infty)^c \in \mathcal{B}(\mathbb{R})$, $[a, \infty) = \{a\} \cup (a, \infty) \in \mathcal{B}(\mathbb{R})$,
  $(-\infty, a) = [a, \infty)^c \in \mathcal{B}(\mathbb{R})$

**Example 1.7 (Borel field on a function space)**

$D(\mathbb{R})$**:** Set of functions which are right-continuous with left limits on $\mathbb{R}$

**Borel field** $\mathcal{B}(D(\mathbb{R}))$ is generated by the sets

$$\big\{f \in D(\mathbb{R}) \mid f(x_1) \in (a_1, b_1), f(x_2) \in (a_2, b_2), \dots, f(x_n) \in (a_n, b_n)\big\},$$
$$n \in \mathbb{Z}, x_i \in \mathbb{R}, -\infty < a_i < b_i < +\infty, i = 1, 2, \dots, n.$$