First Report for Fundamentals of Mathematical and Computing Sciences: Applied Mathematical Sciences

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Submit at the report box 5-4 (located on the 3rd floor between the W8-East and W8-West buildings) or by electronic e-mail.

You can choose to do submit Part A, **or** Part B, **or** Part C.

The report should be written in English. After the correction of the report, there will be an adjustment of the grade in order to compensate the difficulty/easiness of Part A, B, or C.

Part A

Answer all the questions below:

1. If $\mathcal{K}$ is a closed convex cone, show that its dual $\mathcal{K}^*$ is also a closed convex cone.

2. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are convex cones such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$, show that $\mathcal{K}_2^* \subseteq \mathcal{K}_1^*$.

3. Give an example of a polyhedron defined by $\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} \leq \boldsymbol{c}\}$ where $\boldsymbol{A}$ and $\boldsymbol{c}$ have elements equal to $0, \pm 1$, but its vertices do not have integer values (coordinates).

4. Consider the optimization problem

$$\left\{ \begin{array}{ll} \text{maximize} & \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x} \in \{-1,1\}^n \end{array} \right.$$

   where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$. Explain and show why we can always can assume without loss of generality that this matrix is symmetric, positive definite, and diagonally dominant, *i.e.*, $Q_{ii} \geq \sum_{j\neq i}^n |Q_{ij}|$.

5. Given the polynomial $p(x_1,x_2) = 2x_1^2 + x_2^2 - 2\sqrt{2}x_1 + x_1x_2 - 4x_2 + 5$, formulate the semidefinite program (SOS$_d$) explicitly as given in page 13 of the lecture notes. That is, determine the linear equality constraints which the symmetric matrix $\boldsymbol{Q}$ should satisfy.

6. Show that
$$\begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \in \mathcal{S}_+^{n+1} \quad \Leftrightarrow \quad \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \in \mathcal{S}_+^n$$
for all $\boldsymbol{X} \in \mathcal{S}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$.

7. Given m points $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_m$ in $\mathbb{R}^n$, we want to find a point $\boldsymbol{x} \in \mathbb{R}^n$ that minimizes the maximum (Euclidean) distance from itself to the points $\boldsymbol{b}_i$ ($i = 1, 2, \ldots, m$), *i.e.*, we want solve the problem
$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \max_{i=1,2,\ldots,m} \|\boldsymbol{x} - \boldsymbol{b}_i\|_2.$$
Formulate this problem as a conic optimization problem involving second-order cones.

Part B

Complete the proof of Theorem 2.3. That is prove the following statement:

Theorem 1.1 If (DCLP) is bounded from above and it is strictly feasible (*i.e.*, $\exists \bar{\boldsymbol{s}} \in \mathrm{int}(\mathcal{K}^*)$ and $\exists \bar{\boldsymbol{y}} \in \mathbb{R}^m$ such that $\bar{\boldsymbol{s}} = \boldsymbol{c} - \mathcal{A}^*(\bar{\boldsymbol{y}})$), then (CLP) is solvable and its optimal value coincides with the one of (DCLP).

You can use the Separation Theorem for Convex Sets, but not the result of Theorem 2.3 itself proved in class.

Part C

You can write the computer code in any language you most like.

Generating the Max-Cut Problem

Given $n \in \mathbb{N}$, generate a graph $G = (V, E)$ (where $|V| = n$ and $E \subseteq V \times V$) with positive weights, that is, $w : E \to \mathbb{R}_{++}$.

You need to explain in details, how you randomly generate the graph, since it will be evaluated.

Notice that if you generate few edges in the graph, it will be disconnected (*i.e.*, there is a pair of vertices of the graph which does not have a path connecting them), and the numerical results will give only obvious results.

Perform the Goemans-Williamson Randomized Algorithm for the Max-Cut Problem

Execute the algorithm described in page 9 for a given MAX$\in \mathbb{N}$. You will need to solve an Semidefinite Program (SDP). You can opt to install it:

- CVXOPT (http://cvxopt.org/)
- SDPA (http://sdpa.sourceforge.net/download.html#sdpa)
- CSDP (https://projects.coin-or.org/Csdp/)
- SDPT3 (http://www.math.nus.edu.sg/~mattohkc/sdpt3.html). It requires the program MATLAB.

- SeDuMi (http://sedumi.ie.lehigh.edu/). It requires the program Octave (freeware) of the program MATLAB.

Or you can create a data file in the format required by the solver and submit the problem electronically to a server by the internet. In this case, you do not need to install the program in your computer.

- SDPA Online (http://sdpa.sourceforge.net/online.html)
- NEOS Server (http://www.neos-server.org/neos/)

Wrap Up in a Report

Most probably, you will not know the optimal value for the randomly generate max-cut problem.

Remark 1.2 If you write a code to find the optimal value for your problem and you could justify that it is really the optimal value, you will get a bonus point.

Therefore, you will only have the upper bound given by the SDP relaxation of the max-cut problem and the possible candidates for the max-cut given by the Goemans-Williamson's algorithm.

For a reasonable large n and MAX, write a report on the quality of the solutions given by the Goemans-Williamson's algorithm. Also check if Theorem 4.1 is valid from the numerical results.

There will be no specific format for the report, but you will be required to write a report with an analytic analysis of the numerical results you obtained by the computations for a graduate level student.