Corollary 2.4 Assume that at least one of the problems (CLP) or (DCLP) is bounded and strictly feasible. Then a primal-dual feasible solution $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}) \in \mathcal{K} \times \mathbb{R}^m \times \mathcal{K}^*$ is optimal to the respective problems if and only if

(a) $\langle \boldsymbol{c}, \boldsymbol{x} \rangle = \langle \boldsymbol{b}, \boldsymbol{y} \rangle$

or

(b) $\langle \boldsymbol{x}, \boldsymbol{c} - \mathcal{A}^*(\boldsymbol{y}) \rangle = 0$

Proof:

If $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s})$ is primal-dual feasible

$$\langle \boldsymbol{c}, \boldsymbol{x} \rangle - \langle \boldsymbol{b}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{c} - \mathcal{A}^*(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{s} \rangle.$$

∎

2.1 Exercises

1. Show that the dual problem of (DCLP) is exactly (CLP) (when $\mathcal{K}$ in fact is a closed convex cone).

2. Complete the proof of Theorem 2.3.

3 Linear Program Relaxation

In the majority of situations, an optimization problem we want to solve is extremely difficult. That happens in both theory and numerical sense.

In this case, we can always try to solve using some heuristic or meta-heuristic approach such as random algorithms, tabu search, simulated annealing, multiple-start, genetic algorithms, *etc.*

In very particular cases when "we are luck", we can obtain a good approximation for the optimal value and/or solution performing a conic linear program relaxation.

The relaxation methods and examples given in this lecture are far from being complete or even do not have a coverage of most important problems in mathematical optimization. However, we will try to detail some of the famous approaches.

We will start with linear program relaxations.

3.1 Totally Unimodular Matrices

Definition 3.1 A matrix $\boldsymbol{A} \in \mathbb{R}^{n \times m}$ is said to be **totally unimodular** if each square submatrix of it has a determinant which is 0, $+1$ or -1. In particular, all of its elements take these values.

Theorem 3.2 ([Schrijver]) Let $\boldsymbol{A}^T \in \mathbb{Z}^{n \times m}$ be a totally unimodular matrix and let $\boldsymbol{c} \in \mathbb{Z}^n$ be an integer vector. Then the polyhedron $\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c}\}$ is equal to the convex hull of integer vectors.

Therefore, we can conclude from Theorem 3.2 that if we have the following integer program

$$\left\{ \begin{array}{ll} \text{maximize} & \boldsymbol{b}^T \boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c} \\ & \boldsymbol{y} \in \mathbb{Z}^m, \end{array} \right.$$

for $\boldsymbol{b} \in \mathbb{Z}^m$, $\boldsymbol{c} \in \mathbb{Z}^n$ and $\boldsymbol{A}^T \in \mathbb{Z}^{n \times m}$ totally unimodular, then solving the following relaxed problem

$$\left\{ \begin{array}{ll} \text{maximize} & \boldsymbol{b}^T \boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T \boldsymbol{y} \leq \boldsymbol{c} \\ & \boldsymbol{y} \in \mathbb{R}^m, \end{array} \right.$$

which can be solved by the simplex method for instance, we obtain the desired solution.