Reporting assignment

Fundamentals of Mathematical and Computing Sciences: Applied Mathematical Science

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Solve the following problems. Post that to the mailbox located on 3rd floor between East and West wings of W8 building.
Due date : August 10th (Monday).

The operator norm $\|\cdot\|_\infty$, the trace norm $\|\cdot\|_{\mathrm{Tr}}$ and the Frobenius norm $\|\cdot\|_F$ are defined as in the lecture:

$$\|A\|_\infty = \sigma_1(A), \quad \|A\|_{\mathrm{Tr}} = \sum_{i=1}^{p} \sigma_i(A), \qquad \|A\|_F = \sqrt{\sum_{i,j} A_{i,j}^2} = \sqrt{\sum_{i=1}^{p} \sigma_i^2(A)},$$

for $A \in \mathbb{R}^{M\times N}$ where $\sigma_i(A)$ is the i-th largest singular value, and $p = \min\{M, N\}$.

1. Prove that
$$\|A\|_{\mathrm{Tr}} = \sup_{A' \in \mathbb{R}^{M\times N}: \|A'\|_\infty \le 1} \langle A, A' \rangle,$$
and, based on this relation, show that
$$\|A + B\|_{\mathrm{Tr}} \le \|A\|_{\mathrm{Tr}} + \|B\|_{\mathrm{Tr}}$$
for all $A, B \in \mathbb{R}^{M\times N}$.
2. Prove that
$$\|A\|_{\mathrm{Tr}} = \frac{1}{2} \min_{U,V: A = UV^\top} \left\{\|U\|_F^2 + \|V\|_F^2\right\} \quad (A \in \mathbb{R}^{M\times N}).$$
You may use the fact that for a symmetric matrix $S \in \mathbb{R}^{d\times d}$ we have $\sum_{j=1}^{k} \lambda_j(S) = \max_{U \in \mathbb{R}^{k\times d}: UU^\top = I} \mathrm{Tr}[USU^\top]$ for $k \le d$ where $\lambda_j(S)$ is the j-th largest eigenvalue of S.
3. Prove that
$$\|A\|_{\mathrm{Tr}}/\sqrt{d} \le \|A\|_F \le \|A\|_{\mathrm{Tr}}$$
for all matrix $A \in \mathbb{R}^{M\times N}$ that has rank d.
4. Prove that the i.i.d. standard normal random variable sequence $g_i \sim N(0,1)$ $(i = 1, \dots, n)$ satisfies
$$\frac{\sqrt{n}}{2} \le \mathrm{E}\left[\sqrt{\sum\nolimits_{i=1}^{n} g_i^2}\right] \le \sqrt{n}.$$
5. Write your idea about an application of low rank matrix estimation that you think is interesting.