4th Lecture

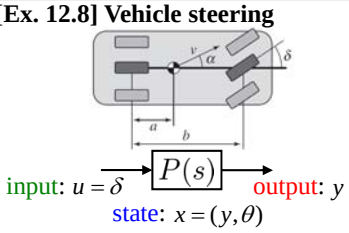
Design Examples

Example 12.8 (Slow Stable Process Zeros) (pp.362 to 364)

Example 12.9 (Fast Stable Process Poles) (pp.364 to 365)

Example 12.10 (Design Rules for Pole Placement) (pp.366 to 368)

[Ex. 12.8] Vehicle steering (Ex. 2.8, 5.12, 7.3, 7.4, 8.6)



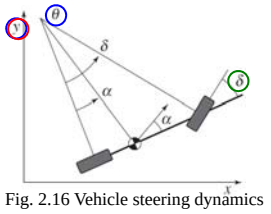


Fig. 2.16 Vehicle steering dynamics

State equation : Plant model (§ 8.3)

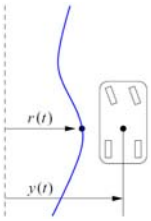
$$\frac{dx}{dt}=Ax+Bu\ ,\quad y=Cx+Du$$

$$A=\begin{bmatrix}0&1\\0&0\end{bmatrix}\quad B=\begin{bmatrix}\gamma\\1\end{bmatrix}\quad C=\begin{bmatrix}1&0\end{bmatrix}\quad D=0$$

$\gamma=0.5$

Transfer function

$$P(s)=C(sI-A)^{-1}B+D=\frac{\gamma s+1}{s^2}$$



[Ex. 12.8] Vehicle steering

Controller

Observer

$$\frac{d\hat{x}}{dt}=A\hat{x}+Bu+L(y-C\hat{x})$$

State feedback

$$u=-K\hat{x}+k_r r$$

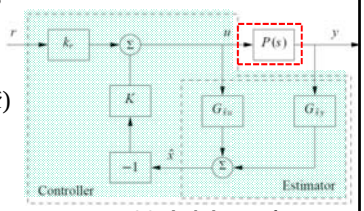


Fig. 8.9 Block diagram for a steering control system

Closed loop system $\tilde{x}=x-\hat{x}$

$$\frac{d}{dt}\begin{bmatrix}x\\\tilde{x}\end{bmatrix}=\begin{bmatrix}A-BK&BK\\0&A-LC\end{bmatrix}\begin{bmatrix}x\\\tilde{x}\end{bmatrix}+\begin{bmatrix}Bk_r\\0\end{bmatrix}r$$

$$y=\begin{bmatrix}C&0\end{bmatrix}\begin{bmatrix}x\\\tilde{x}\end{bmatrix}$$

$$\lambda(s)=\det(sI-A+BK)\det(sI-A+LC)$$

Separation Principle

[Ex. 12.8] Vehicle steering **Pole Placement**

State feedback gain

$$\det(sI-A+BK)=s^2+(\gamma k_1+k_2)s+k_1$$
$$=s^2+2\zeta_c\omega_c+\omega_c^2$$

$$k_1=\omega_c^2\quad k_2=2\zeta_c\omega_c-\gamma\omega_c^2$$

Observer gain

$$\det(sI-A+LC)=s^2+l_1s+l_2$$
$$=s^2+2\zeta_o\omega_o s+\omega_o^2$$

$$l_1=2\zeta_o\omega_o\quad l_2=\omega_o^2$$

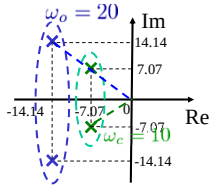
Faster close loop system

$$\begin{cases}\omega_c=10 & \omega_0=20\\ \zeta_c=0.707 & \zeta_o=0.707\end{cases}$$

$-7.07\pm7.07i\quad -14.14\pm14.14i$

Should be GOOD?

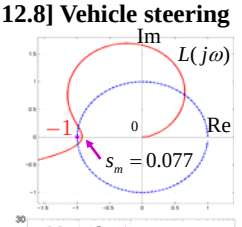
$$\begin{cases}k_1=100 & l_1=28.28\\ k_2=-35.86 & l_2=400\end{cases}$$



× eig. val. of A - BK

× eig. val. of A - LC

[Ex. 12.8] Vehicle steering



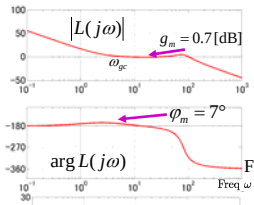
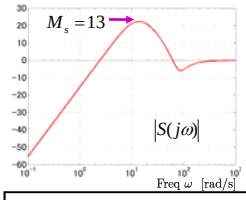


Fig. 12.11



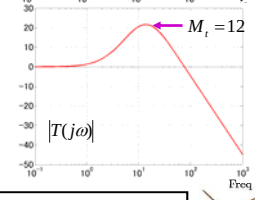


Fig. 12.12

The controller achieves a good control performance, doesn't it?

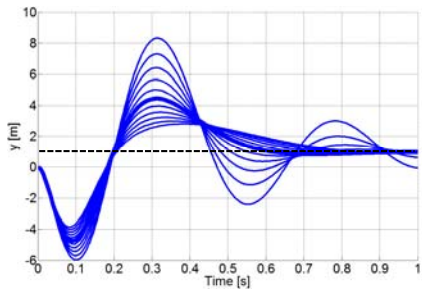
[Ex. 12.8] Vehicle steering

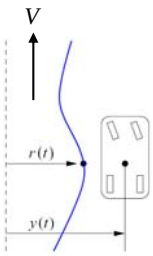
$$\begin{cases}\omega_c=10 & \omega_0=20\\ \zeta_c=0.707 & \zeta_o=0.707\end{cases}$$

$-7.07\pm7.07i\quad -14.14\pm14.14i$

Nominal $V=1$

Variation $0.95\le V\le1.05$

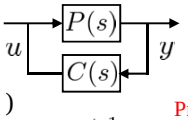




Oops・・・ both performance and robust stability are poor !

[Ex. 12.8] Slow Stable Process Zeros

What happens ?



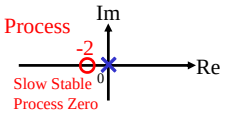
Process (from u to y)

$$P(s) = C(sI - A)^{-1}B + D = \frac{\gamma s + 1}{s^2}$$

$$= \frac{0.5s + 1}{s^2}$$

Pole : $p = 0, 0$

Zero : $z = -2$



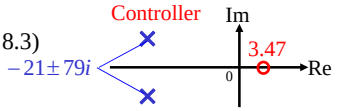
Controller (from y to u) (§ 8.3)

$$C(s) = \frac{KG_{\hat{x}y}(s)}{1 + KG_{\hat{x}u}(s)}$$

$$= \frac{-11516s + 40000}{s^2 + 42.4s + 6657.9}$$

Pole : $p = -21 \pm 79i$

Zero : $z = 3.47$



× Pole
○ Zero

[Ex. 12.8] Slow Stable Process Zeros

$$T = \frac{PC}{1 + PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

$P = \frac{n_p}{d_p}$, $C = \frac{n_c}{d_c}$

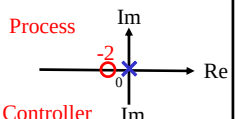
T has the poles of closed-loop system and its zeros are given by zeros of the process and controller

Process (from u to y)

$$P(s) = \frac{0.5s + 1}{s^2} \leftarrow n_P$$

Pole : $p = 0, 0$

Zero : $z = -2$

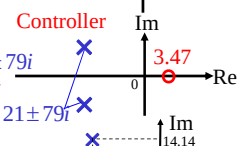


Controller (from y to u)

$$C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6657.9} \leftarrow n_c$$

Pole : $p = -21 \pm 79i$

Zero : $z = 3.47$



Closed –loop (from r to y)

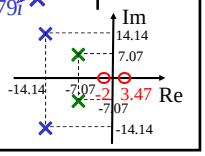
$$T(s) = \frac{-5758(s+2)(s-3.47)}{(s^2 + 14.14s + 100)(s^2 + 28s + 400)}$$

Pole : $d_p d_c + n_p n_c = 0$

$p = -7.07 \pm 7.07i$

$-14.14 \pm 14.14i$

Zero : $z = -2, 3.47$



[Ex. 12.8] Slow Stable Process Zeros

$$T = \frac{PC}{1 + PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

closed loop poles **Pole :** $-7.07 \pm 7.07i$ $(\omega_c = 10)$

$-14.14 \pm 14.14i$ $(\omega_o = 20)$

Zero : $-2, 3.47$

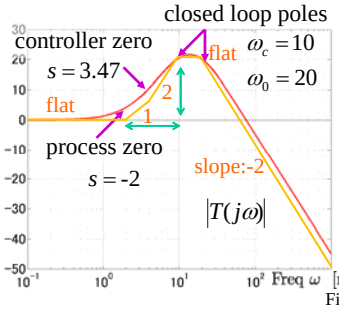
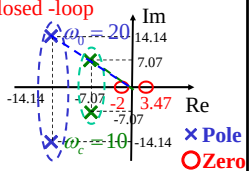


Fig. 12.12

Closed -loop



× Pole
○ Zero

The magnitude of the peak depends on the ratio of the zeros and the poles of the transfer function.

assign a closed loop pole close to the slow process zero

[Ex. 12.8] Slow Stable Process Zeros

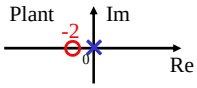
Process

$$P(s) = \frac{0.5s + 1}{s^2}$$

Pole : $p = 0, 0$

Zero : $z = -2$

*process stable zero $z = -2$



Faster close loop system

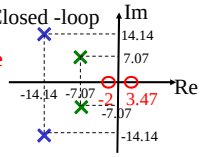
$$\begin{cases} \omega_c = 10 & \omega_0 = 20 \\ \zeta_c = 0.707 & \zeta_o = 0.707 \end{cases} \rightarrow$$

closed loop pole

$-7.07 \pm 7.07i$

$-14.14 \pm 14.14i$

Closed -loop



Assign a closed loop pole close to the slow process zero.

$$\begin{cases} \omega_c = 10 & \omega_0 = 20 \\ \zeta_c = 2.6 & \zeta_o = 0.707 \end{cases} \rightarrow$$

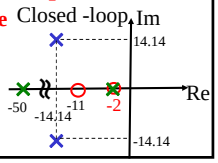
closed loop pole

$-2, -50$

$-14.14 \pm 14.14i$

$$\left(\begin{aligned} \det(sI - A + BK) &= s^2 + 2 \cdot 2.6 \cdot 10s + 10^2 \\ &= (s+2)(s+50) \end{aligned} \right)$$

Closed -loop

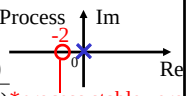


[Ex. 12.8] Slow Stable Process Zeros

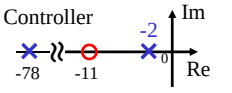
Process: $P(s) = \dfrac{0.5s + 1}{s^2}$

$$\begin{cases} \omega_c = 10 & \omega_0 = 20 \\ \zeta_c = 2.6 & \zeta_o = 0.707 \end{cases} \rightarrow C(s) = \frac{3628(s + 11.02)}{(s+2)(s+78.28)}$$

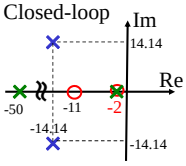
*process stable zero

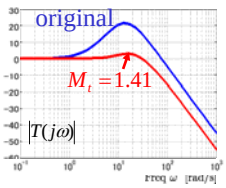


$$T(s) \approx \frac{1814(s+11)(s+2)}{(s+2)(s+50)(s^2 + 28s + 400)}$$



Closed-loop





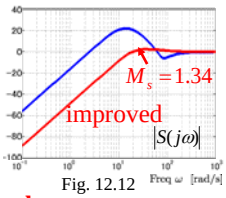


Fig. 12.12

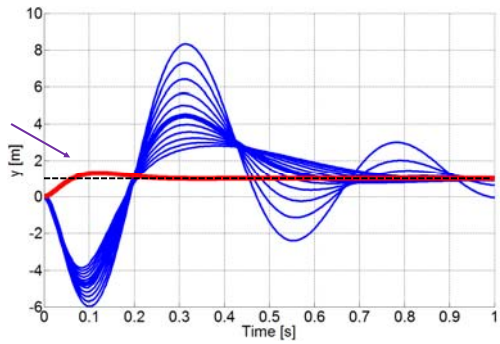
Assign a closed loop pole close to the slow process zero

[Ex. 12.8] Slow Stable Process Zeros

$$\begin{cases} \omega_c = 10 & \omega_0 = 20 \\ \zeta_c = 0.707 & \zeta_o = 0.707 \end{cases}$$

$$\begin{cases} \omega_c = 10 & \omega_0 = 20 \\ \zeta_c = 2.6 & \zeta_o = 0.707 \end{cases}$$

2DOF controller



[Ex. 12.9] Fast Stable Process Poles

Process

$$P(s)=\frac{b}{s+a}$$

Pole : $p=-a$

PI controller

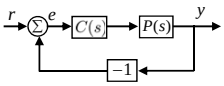
$$C(s)=\frac{k_p s+k_i}{s}$$

Pole : $p=0$

Zero : $z=-k_i/k_p$

Loop transfer function

$$L(s)=\frac{b(k_p s+k_i)}{s(s+a)}$$



Closed loop characteristic polynomial

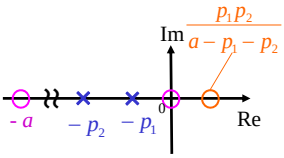
$$s(s+a)+b(k_p s+k_i)=s^2+(a+bk_p)s+k_i b$$

desired closed loop poles : $-p_1 \quad -p_2 \quad s^2+(p_1+p_2)s+p_1p_2$

$$k_p=\frac{p_1+p_2-a}{b} \qquad k_i=\frac{p_1p_2}{b}$$

Complementary sensitivity

$$T(s)=\frac{(p_1+p_2-a)s+p_1p_2}{(s+p_1)(s+p_2)}$$



Sensitivity

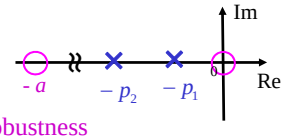
$$S(s)=\frac{s(s+a)}{(s+p_1)(s+p_2)}$$

assume $p_1<p_2 \ll a$

[Ex. 12.9] Fast Stable Process Poles

assume $p_1<p_2 \ll a$

$$S(s)=\frac{s(s+a)}{(s+p_1)(s+p_2)}$$



poor robustness

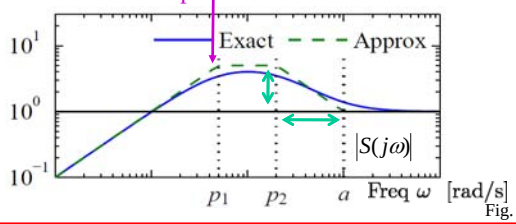


Fig. 12.13

Choose one closed loop pole equal to the process pole

$p_2=a$

[Ex. 12.9] Fast Stable Process Poles

Process

$$P(s)=\frac{b}{s+a}$$

Pole : $p=-a$

PI controller $C(s)=\frac{k_p s+k_i}{s}$

$$k_p=\frac{p_1+p_2-a}{b}, \quad k_i=\frac{p_1p_2}{b}$$

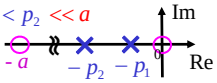
Pole : $p=0$

Zero : $z=-k_i/k_p$

Sensitivity

$$S(s)=\frac{s(s+a)}{(s+p_1)(s+p_2)}$$

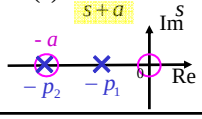
$p_1<p_2 \ll a$



Choose one closed loop pole equal to the process pole.

$p_2=a \quad k_p=\frac{p_1}{b} \quad k_i=\frac{ap_1}{b} \quad C(s)=\frac{k_p(s+a)}{s}$ **Pole :** $p=0$ **Zero :** $z=-a$

$$L(s)=\frac{b}{s+a}\cdot\frac{k_p(s+a)}{s}=\frac{bk_p}{s} \qquad S(s)=\frac{s}{s+bk_p} \qquad T(s)=\frac{bk_p}{s+bk_p}$$



The fast process pole is canceled by a controller zero

[Ex. 12.9] Fast Stable Process Poles

Process

$$P(s)=\frac{b}{s+a}$$

Pole : $p=-a$

PI controller $C(s)=\frac{k_p(s+a)}{s}$

$$k_p=\frac{p_1}{b} \qquad k_i=\frac{ap_1}{b}$$

Pole : $p=0$

Zero : $z=-a$

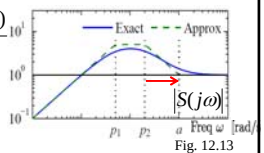
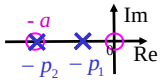
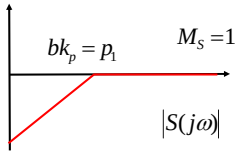


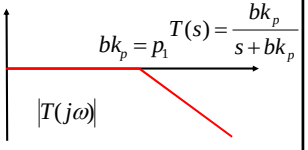
Fig. 12.13

Sensitivity

$$S(s)=\frac{s(s+a)}{(s+p_1)(s+p_2)} \xrightarrow{p_2=a} S(s)=\frac{s}{s+bk_p}$$





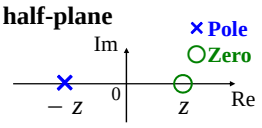


Choose one closed loop pole equal to the process pole

Recap. [Ex. 11.7] Zero in the right half-plane

All-pass system with a RHP zero

$$P_{ap}(s)=\frac{z-s}{z+s} \qquad z>0$$

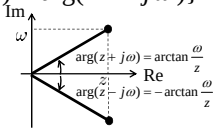


*RHP : right half-plane

Phase lag of the all-pass system

$$-\arg P_{ap}(j\omega)=-\{\arg(z-j\omega)-\arg(z+j\omega)\}$$

$$=2\arctan\frac{\omega}{z}$$



gain crossover frequency inequality

$$-\arg P_{ap}(j\omega_{gc}) \le \pi-\varphi_m+n_{gc}\frac{\pi}{2}=:\varphi_l \quad (11.15)$$

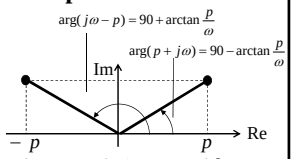
Bound on the crossover frequency ω_{gc}

$$\omega_{gc}<z\tan(\varphi_l/2) \quad (11.16)$$

Recap. [Ex. 11.8] Pole in the right half-plane

All-pass system with a RHP pole

$$P_{ap}(s)=\frac{s+p}{s-p} \qquad p>0$$



Phase lag of the all-pass system

$$-\arg P_{ap}(j\omega)=-\{\arg(p+j\omega)-\arg(j\omega-p)\}$$

$$=2\arctan\frac{p}{\omega}$$

gain crossover frequency inequality

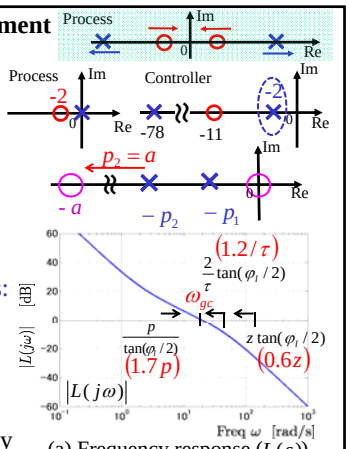
$$-\arg P_{ap}(j\omega_{gc}) \le \pi-\varphi_m+n_{gc}\frac{\pi}{2}=:\varphi_l \quad (11.15)$$

Bound on the crossover frequency ω_{gc}

$$\omega_{gc}>\frac{p}{\tan(\varphi_l/2)} \quad (11.17)$$

Design Rules for Pole Placement

- Slow stable process zeros: should be cancelled by the controller poles.
- Fast stable process poles: should be cancelled by the controller zeros
- Slow unstable process zeros: achievable ω_{gc} must be smaller than the slowest unstable process zero.
- Fast unstable process poles: the gain crossover frequency must be sufficiently large.



(a) Frequency response ($L(s)$)

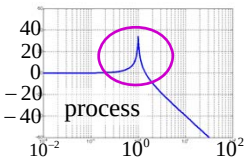
Fig. 11.8

[Ex. 12.10] Nanopositioning system for an atomic force microscope

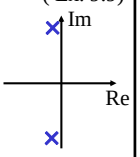
$$P(s) = \frac{a^2}{s^2 + 2\zeta a s + a^2}$$

$\zeta = 0.01, \quad a = 1$

poles: $-0.0100 \pm 0.9999j$



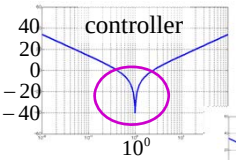
(Ex. 9.9)

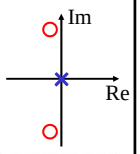


Fast stable process poles should be canceled by the controller zeros

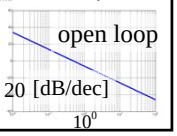
PID Controller

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s} = \frac{k_i}{s}\frac{s^2 + 2\zeta a s + a^2}{a^2}$$





$$\rightarrow \begin{cases} k_p = \dfrac{2\zeta k_i}{a} \\ k_d = \dfrac{k_i}{a^2} \end{cases} \quad k_i = 0.5 \longrightarrow L = \frac{0.5}{s}$$



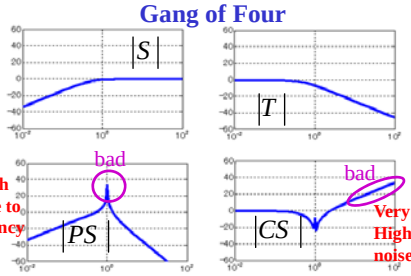
[Ex. 12.10] Nanopositioning system for an atomic force microscope

(Ex. 9.9)

$$P(s) = \frac{a^2}{s^2 + 2\zeta a s + a^2} \qquad C(s) = \frac{k_i}{s}\frac{s^2 + 2\zeta a s + a^2}{a^2}$$

$$L(s) = \frac{0.5}{s} \longrightarrow S(s) = \frac{s}{s + 0.5} \qquad T(s) = \frac{0.5}{s + 0.5}$$

Gang of Four



Sensitive to load disturbances with frequencies close to resonant frequency

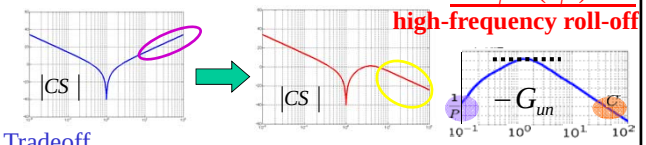
Very sensitive to High-frequency noise

Fig. 12.14

[Ex. 12.10] Nanopositioning system for an atomic force microscope

$$C(s) = \frac{k_i}{s}\frac{s^2 + 2\zeta a s + a}{a^2} \rightarrow C(s) = \frac{k_i}{s}\frac{s^2 + 2\zeta a s + a}{a^2}\frac{1}{1 + T_f s + (T_f s)^2/2}$$

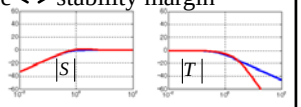
high-frequency roll-off

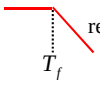


Tradeoff

attenuation of measurement noise ↔ stability margin

Big T_f
- reduces the effects of noise
- reduces the stability margin





reasonable choice

$$T_f = \frac{0.2}{k_i}$$

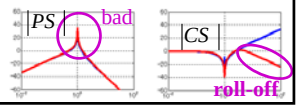


Fig. 12.14

[Ex. 12.10] Nanopositioning system for an atomic force microscope

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s} \qquad P(s) = \frac{a^2}{s^2 + 2\zeta a s + a^2}$$

Loop transfer function

$$L(s) = \frac{a^2(k_d s^2 + k_p s + k_i)}{s(s^2 + 2\zeta a s + a^2)}$$

Characteristic polynomial

$$s^3 + (a^2 k_d + 2\zeta a)s^2 + (k_p + 1)a^2 s + k_i a^2$$

general third-order parameterized polynomial

$$s^3 + (\alpha_0 + 2\xi)\omega_0 s^2 + (1 + 2\alpha_0\xi)\omega_0^2 s + \alpha_0\omega_0^3$$

α_0, ξ : relative configuration of the poles

ω_0 : magnitudes of the poles

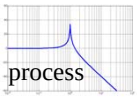
$$k_p = \frac{(1 + 2\alpha_0\xi)\omega_0^2}{a^2} - 1, \quad k_i = \frac{\alpha_0\omega_0^3}{a^2}, \quad k_d = \frac{(\alpha_0 + 2\xi)\omega_0 - 2\zeta a}{a^2}$$

[Ex. 12.10] Nanopositioning system for an atomic force microscope

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s(1 + sT_f + (sT_f)^2/2)} \qquad P(s) = \frac{a^2}{s^2 + 2\zeta a s + a^2}$$

good filtering time constant : $T_f = 0.1\dfrac{k_d}{k_i}$

parameters : $\xi = 0.707 \quad \alpha_0 = 1 \quad \omega_0 = 1, 2, 4$



Controller

$\omega_0 = 1$

Poles: $0, -4.2 \pm 4.2i$

Zeros: $-0.3 \pm 0.6i$

$\omega_0 = 2$

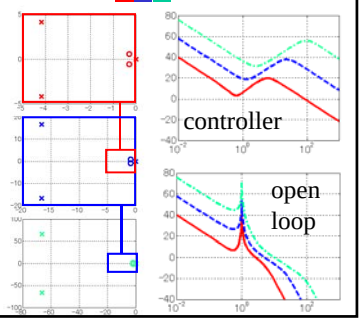
Poles: $0, -2.0 \pm 1.7i$

Zeros: $-0.9 \pm 0.9i$

$\omega_0 = 4$

Poles: $0, -66.4 \pm 66.4i$

Zeros: $-16.6 \pm 16.6i$



Gang of Four

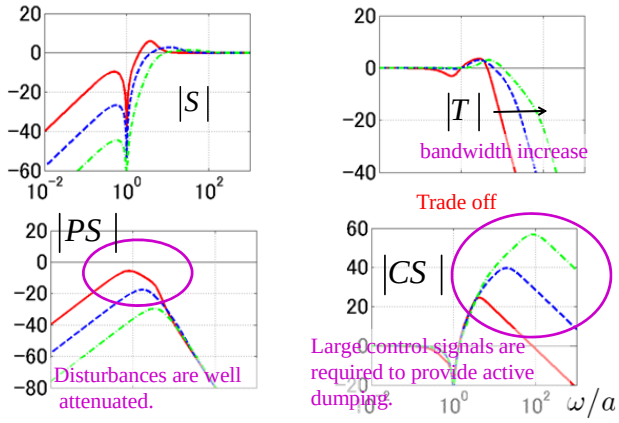


Fig. 12.15

4th Lecture

Design Examples

Example 12.8 (Slow Stable Process Zeros) (pp.362 to 364)

Example 12.9 (Fast Stable Process Poles) (pp.364 to 365)

Example 12.10 (Design Rules for Pole Placement) (pp.366 to 368)

Next: 5th Lecture

11 Frequency Domain Design

11.5 Fundamental Limitations (pp.331 to 340)

Keyword : Right Half-Plane Poles and Zeros
Gain Crossover Frequency Inequality