Basic Mathematics

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1 Logic

$\wedge$ denotes "and"

$\vee$ denotes "or"

$\neg$ denotes "not"

$p \Rightarrow q$ denotes "if p then q"

$p \Leftarrow q$ denotes "if q then p"

$p \Leftrightarrow q$ is defined by $(p \Rightarrow q) \wedge (q \Leftarrow p)$ and is read "p if and only if (iff) q"

$p :\Leftrightarrow q$ is read "p is defined by q", and implies $p \Leftrightarrow q$.

$x := y$ is read "x is defined by y, and implies $x = y$.

$\forall$ denotes "for all"

$\exists$ denotes "exists"

2 Sets and Functions

Definition 2.1 (Power Set). *The power set of set X is the set of all subsets of X denoted*

$$\mathcal{P}(X) := \{A | A \subset X\}$$

Definition 2.2 (Binary Relation). *A binary relation R between an element in set X and an element in Y is a subset of the Cartesian product $X \times Y$, that is $R \subset X \times Y$.*

The statement $(x, y) \in R$ is read "x is R-related to y" and is denoted xRy.

When $X = Y$, binary relation $R \subset X^2$ is said to be defined on set X.

Definition 2.3 (Function). *A function $f : X \to Y$ is a binary relation $f \subset X \times Y$ that associates to each element $x \in X$ exactly one element $y \in Y$, that is:*

- $\forall x \in X \exists y \in Y, (x, y) \in f$
- $\forall x \in X \forall y, y' \in Y, [(x, y), (x, y') \in f \Rightarrow y = y']$

$(x, y) \in f$ *is denoted $y = f(x)$.*

Definition 2.4 (Image and Preimage (Inverse Image)). *Let $f : X \to Y$ be a function.*

Image $\forall A \subset X,\ f(A) := \{f(x) | x \in A\}$

Preimage $\forall B \subset Y,\ f^{-1}(B) := \{x \in X | f(x) \in B\}$

3 Vectors

Following notations are used for vectors and cartesian products. Particularly, vectors are denoted with normal fonts.

- $x = (x_i)_{i \in N} = (x_1, \ldots, x_N) \in X = \times_{i \in N} X_i$
- $x_{-i} := (x_j)_{j \in N \setminus \{i\}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{-i} := \times_{j \in N \setminus \{i\}} X_j$

4 Real Number and its Cartesian Products

Definition 4.1. *Denote $\Re$ the set of real numbers and $\Re_+$ the set of nonnegative real numbers.*

Definition 4.2 (Maximization (Minimization))**.** *Let $f : X \to \Re$ be a real-valued function and $A \subset X$. For minimization, replace* max *with* min*.*

- $\arg\max_{x \in A} f(x) := \{x \in A \mid \forall y \in A, f(y) \leq f(x)\}$
- $\max_{x \in A} f(x) := \max f(A)$

Note that $\arg\max_{x \in A} f(x)$ need not be a singleton set, whereas $\max_{x \in A} f(x)$ is a single maximum element (if one exists) in $f(A)$.

Definition 4.3 (Product Order)**.** *For $x, y \in \Re^N$:*

- $x \geq y :\Leftrightarrow \forall i \in N, x_i \geq y_i$
- $x > y :\Leftrightarrow x \geq y \land x \neq y$
- $x \gg y :\Leftrightarrow \forall i \in N, x_i > y_i$

Definition 4.4 (Pareto efficiency (Pareto optimality))**.** *$x \in S \subset \Re^N$ is*

(weakly) Pareto efficient *iff $\neg \exists y \in S, y \gg x$*

strongly Pareto efficient *iff $\neg \exists y \in S, y > x$*

5 Probability

Definition 5.1. *Denote $\Delta(X)$ a set of probability distributions over set X.*
If X is finite, $\Delta(X)$ is called a simplex and fulfills

$$\Delta(X) = \{\phi \in \Re^X | \sum_{x \in X} \phi(x) = 1 \land (\forall x \in X, \phi(x) \geq 0)\}$$

Definition 5.2 (Support)**.** *Support of a probability distribution $\phi \in \Delta(X)$ is*

$$\text{supp } \phi = \{x \in X | \phi(x) \neq 0\}$$

Definition 5.3 (Restriction)**.** *Probability $\phi \in \Delta(X)$ is restricted to $Y \subset X$ iff $\text{supp } \phi \subset Y$.*