Chapter 8 Network Models

Based on Graph Theory

1. **<u>Shortest path problems</u>** (最短路問題)
2. Maximum flow problems (最大フロー問題)
3. **<u>CPM-PERT project-scheduling models</u>** (工程計画法の一つ)
4. Minimum-cost network flow problems (最小費用フロー問題)

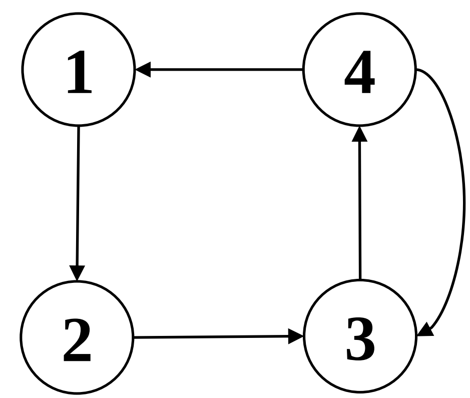
Basic Definition

Node: is vertices of a graph or network. Initial node and terminal node in the arc.

Arc (edge/link): consists of an ordered pair of vertices and represent a possible <u>direction</u> of motion.

Chain: is a sequence of arcs that every arc has exactly one vertex in common with the previous arc.

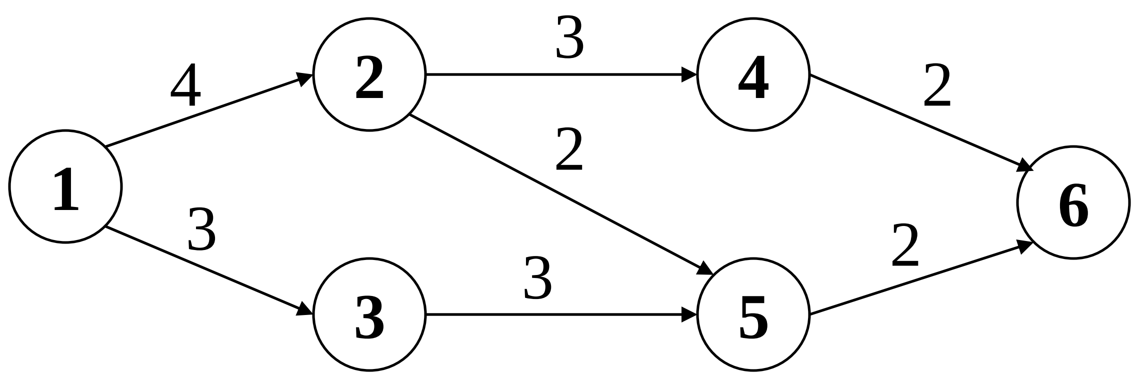
Path: is a chain in which the terminal node of each arc is identical to the initial node of the next arc.



directed graph
(有向グラフ)

8.2 Shortest Path Problems

距離（費用，時間）が最小となる経路を求める問題



Find minimum possible distance (mile or km) from node 1 to node 6

Path A 1 > 2 > 4 > 6 : 9

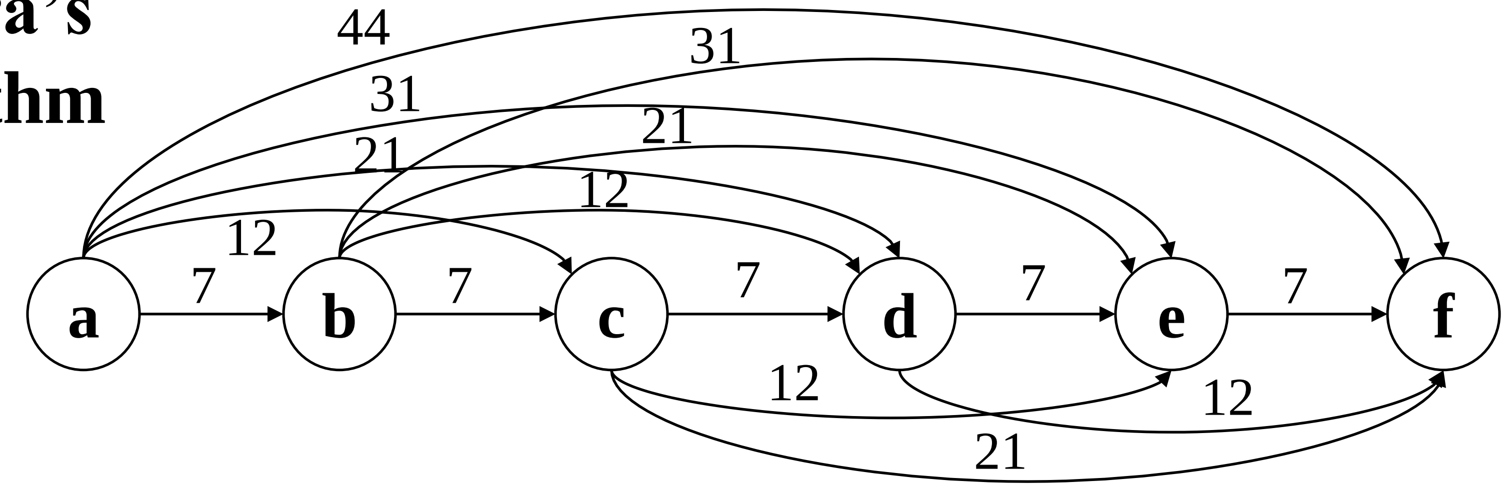
Path B 1 > 2 > 5 > 6 : 8 Shortest

Path C 1 > 3 > 5 > 6 : 8 Shortest

Dijkstra’s Algorithm： ダイクストラ法（1959年）

カーナビゲーション経路探索，鉄道経路探索に実際に活用

Dijkstra's Algorithm



	a*	b	c	d	e	f
d		7	12	21	31	44
t	0	7	12	21	31	44
n		a	a	a	a	a

	a*	b*	c	d	e	f
d			14	19	28	38
t	0	7	12	19	28	38
n		a	a	b	b	b

	a*	b*	c*	d	e	f
d				19	24	33
t	0	7	12	19	24	33
n		a	a	b,c	c	c

	a*	b*	c*	d*	e	f
d					26	31
t	0	7	12	19	24	31
n		a	a	b,c	c	d

	a*	b*	c*	d*	e*	f
d						31
t	0	7	12	19	24	31
n		a	a	b,c	c	d,e

Shortest Path

a – b – d – f, 31

a – c – d – f, 31

a – c – e – f, 31

8.3 Maximum Flow Problems (最大フロー問題)

ある地点から他の地点に物資・資源を送るとき, いくつかの地点を経由し,分岐や合流をしながら物資が送られるとする。送られる過程で各地点から各地点に一度に送ることのできる量の最大値が決まっているとき, 全体で一度に送ることのできる最大量を求める問題。送られる量をフロー(Flow), 出発点をソース(Source), 終着点をシンク(Sink)と呼ぶ。

例：物資輸送，送電網，水道管など

8.5 Minimum-cost Network Flow Problems (最小費用フロー問題)

各リンク(arc, edge[枝])の容量と費用および各ノードの供給量or需要量が与えられたとき, 各リンクの容量を超えず, 各ノードでの流量が供給量or需要量と等しくなるリンクの流れをフローと呼ぶ. このとき，各リンクの流量に対する費用の総和を最小にするフローを求める問題。

7章・8章の輸送問題，最短路問題，最大フロー問題，CPM-PERTなどは最小費用フロー問題の特殊形として表現されたもの.

8.4 CPM and PERT

Model to aid the decision of “Project Scheduling”

CPM: Critical Path Method

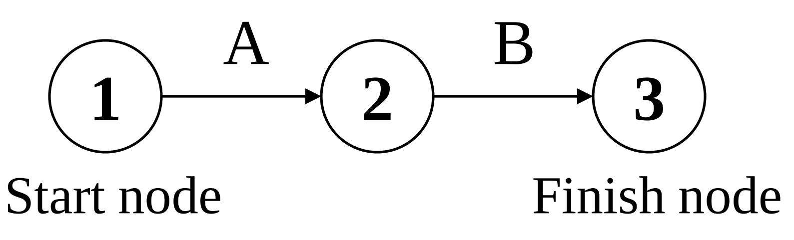
to determine the length of time required to complete a project

PERT: Program Evaluation and Review Technique

to estimate the probability that a project will be completed by a given deadline

AOA (activity on arc):

Project network or Project diagram (Arrow diagram)



Arc (A, B): Activity

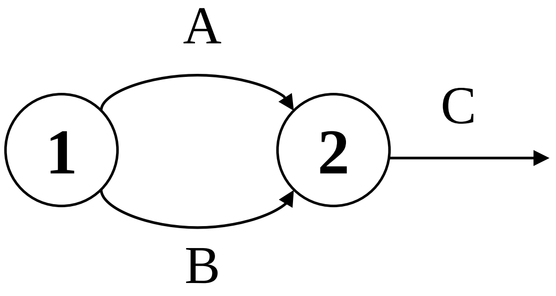
Node (1,2,3):
Completion of activity (Event)

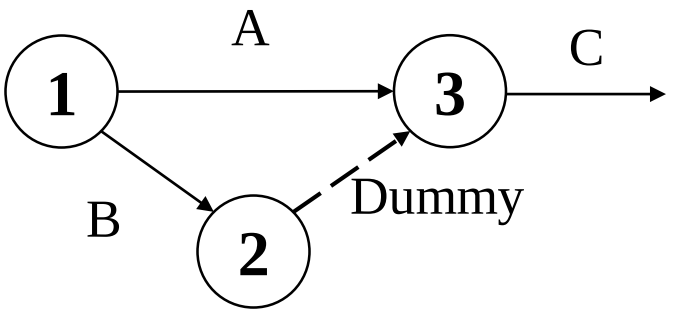
A is a Predecessor of B

Dummy activity

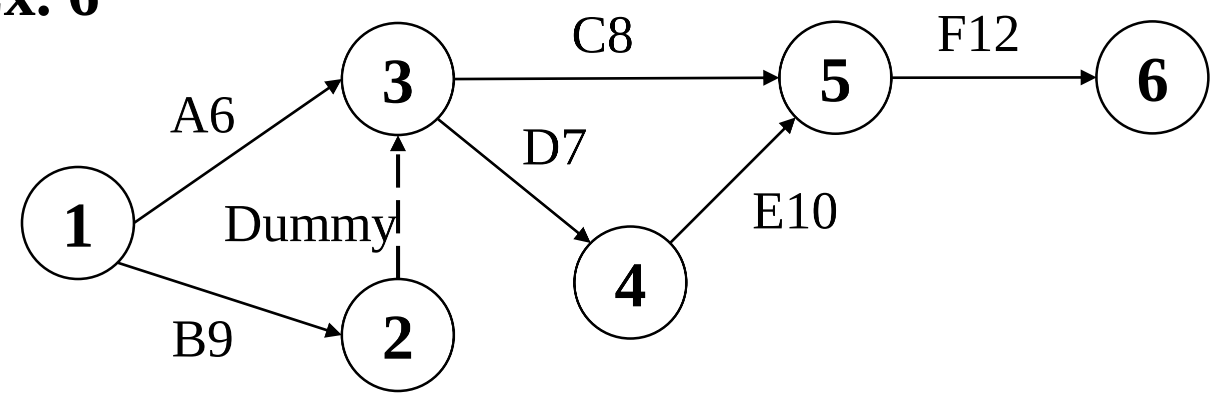
Important rule of AOA

4. An activity should not be represented by more than one arc in the network.
5. Two nodes can be connected by at most one arc.



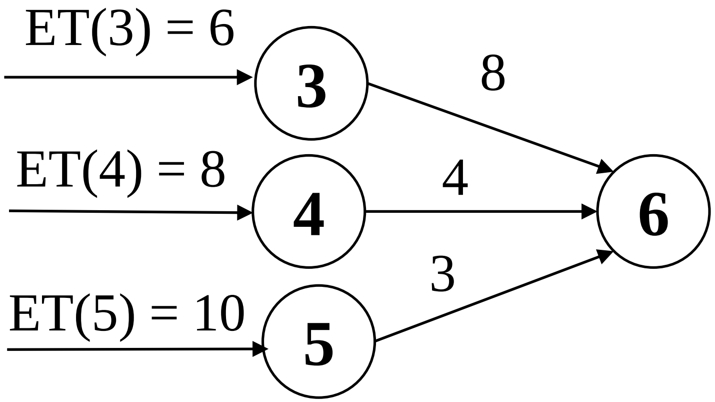


Ex. 6



Early Event Time (ET)

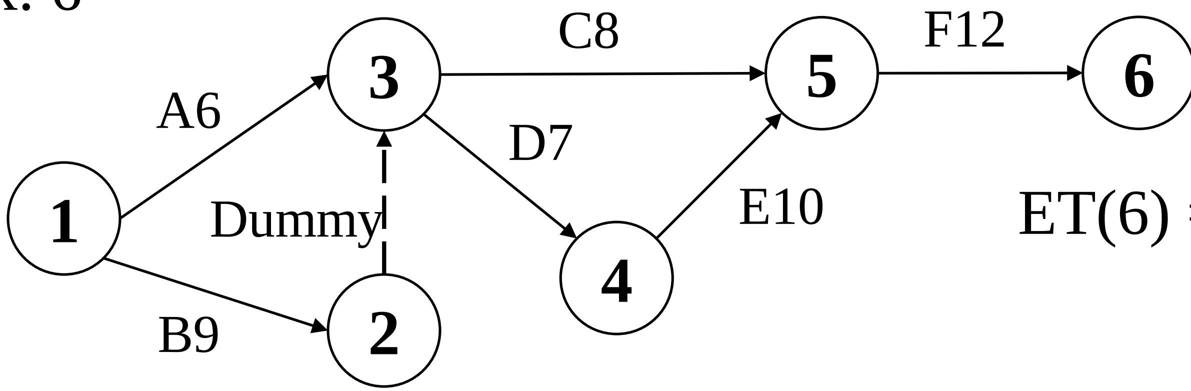
Earliest time at which the event corresponding to node *i* can occur



$$ET(6) = \max \left\{ \begin{array}{l} 6 + 8 = 14 \\ 8 + 4 = 12 \\ 10 + 3 = 13 \end{array} \right\} = 14$$

$ET(1) = 0$

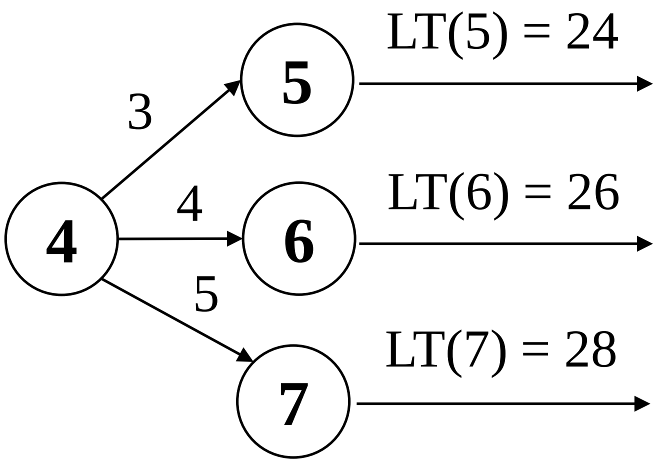
Ex. 6



$ET(6) = 38$

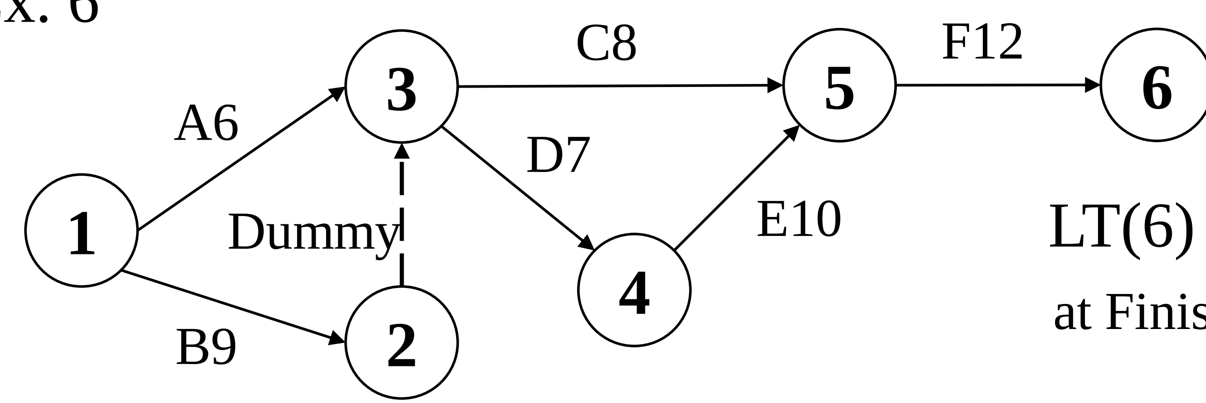
Late Event Time (LT)

Latest time at which the event corresponding to node *i* can occur without delaying the completion of the project



$$LT(4) = \min \left\{ \begin{array}{l} 24 - 3 = 21 \\ 26 - 4 = 22 \\ 28 - 5 = 23 \end{array} \right\} = 21$$

Ex. 6



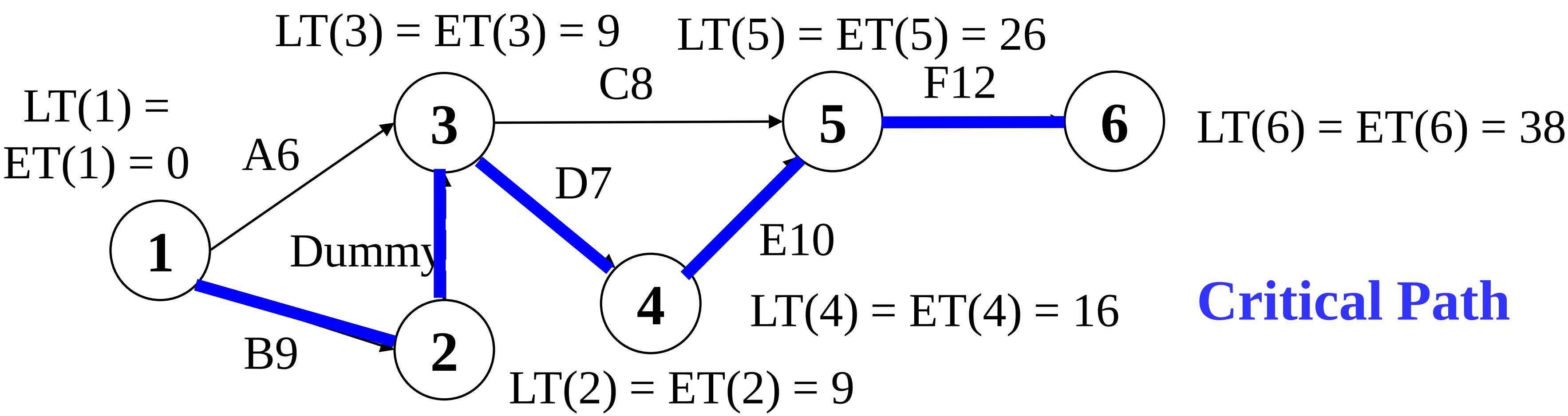
LT(6) = ET(6) = 38

at Finish node

Total Float (TF) and Critical Path

Total Float is the amount by which the starting time of activity (i, j) could be delayed beyond its earliest possible starting time without delaying the completion of the project (assuming no other activities are delayed).

$$TF(i, j) = LT(j) - ET(i) - t_{ij}$$



A TF(1,3) = 3, D TF(3,4) = 0

B TF(1,2) = 0, E TF(4,5) = 0

C TF(3,5) = 9, F TF(5,6) = 0

Dummy TF(2,3) = 0

Critical Activity is any activity with a total float of zero

Critical Path is a path from 1 to the finish node that consist entirely of critical activity

Free Float (FF)

Free Float is the amount by which the starting time of the activity corresponding to arc (i, j) (or the duration of the activity) can be delayed without delaying the start of any later activity beyond its earliest possible starting time.

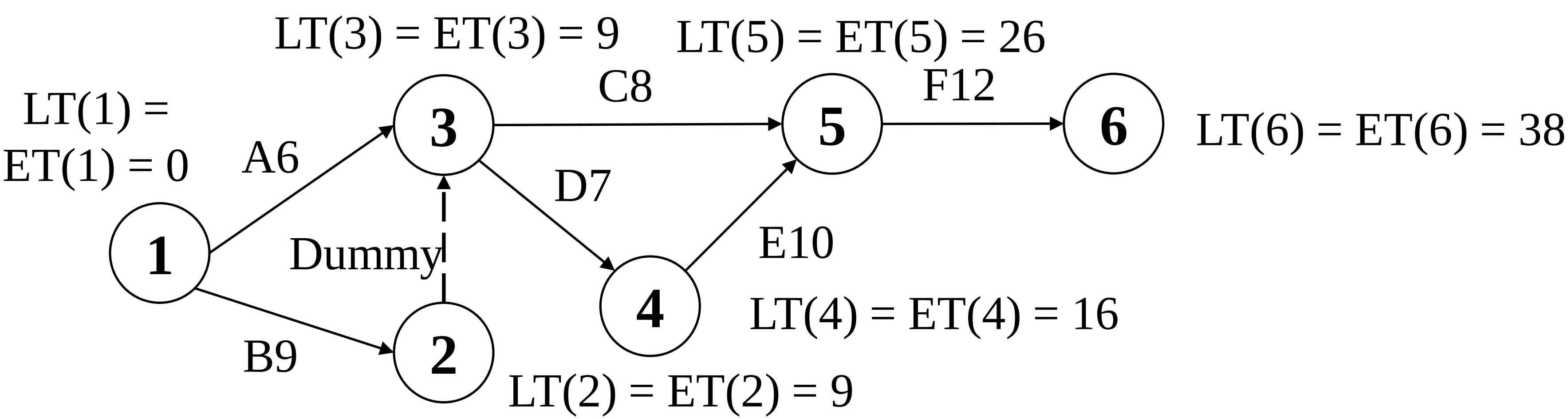
$$FF(i, j) = ET(j) – ET(i) - t_{ij}$$

A FF(1,3) = 3, D FF(3,4) = 0

B FF(1,2) = 0, E FF(4,5) = 0

C FF(3,5) = 9, F FF(5,6) = 0

Since the free float for activity C is 9 days, a delay in the start of activity C or a delay in the duration of activity C of more than 9 days will delay the start of some later activity F.



PERT: Program Evaluation and Review Technique

<u>Modeling the duration of each activity as a random variable</u>

a = estimate of the activity's duration under the most favorable conditions

b = estimate of the activity's duration under the least favorable conditions

m = most likely value for the activity's duration

T_{ij} = duration of activity (i,j)

$$E(T_{ij}) = \frac{a + 4m + b}{6} \qquad Var(T_{ij}) = \frac{(b-a)^2}{36}$$

under assumption that T_{ij} follows a beta distribution

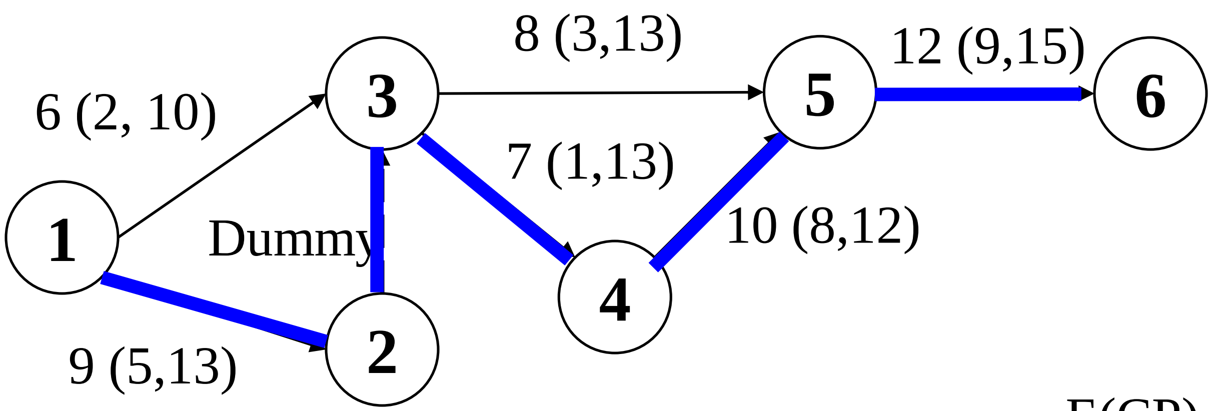
Duration of all activities are independent

$$CP = \sum_{(i,j) \in CP} T_{ij}$$ is normal distribution

Test the probability of the duration X of the project completion

$$Z = \frac{\mathrm{X} - E(\mathrm{CP})}{\sqrt{\mathrm{VarCP}}}$$

Random variable, m (a, b)



LT(6) = ET(6) = 38

Critical Path

E(CP) = 9+0+7+10+12=38

Var(CP)
= 1.78+0+4+0.44+1=7.22

SD (CP) = 2.69

$$E(T_{12}) = \frac{5+13+36}{6} = 9 \qquad Var(T_{12}) = \frac{(13-5)^2}{36} = 1.78$$

$$E(T_{13}) = \frac{2+10+24}{6} = 6 \qquad Var(T_{13}) = \frac{(10-2)^2}{36} = 1.78$$

$$E(T_{35}) = \frac{3+13+32}{6} = 8 \qquad Var(T_{35}) = \frac{(13-3)^2}{36} = 2.78$$

$$E(T_{34}) = \frac{1+13+28}{6} = 7 \qquad Var(T_{34}) = \frac{(13-1)^2}{36} = 4.00$$

$$E(T_{45}) = \frac{8+12+40}{6} = 10 \qquad Var(T_{45}) = \frac{(12-8)^2}{36} = 0.44$$

$$E(T_{56}) = \frac{9+15+48}{6} = 12 \qquad Var(T_{56}) = \frac{(15-9)^2}{36} = 1.00$$

X = 35 days?

$$Z = \frac{35-38}{2.69} = -1.12$$

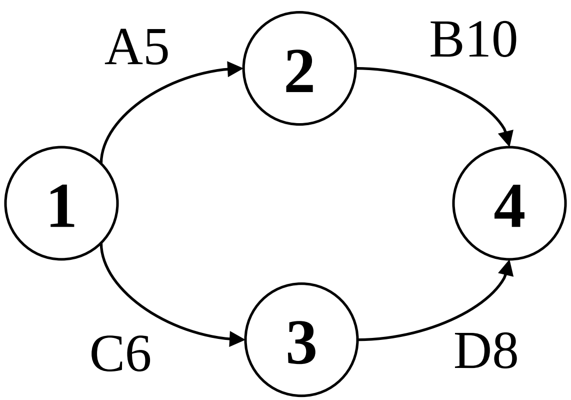
$\Phi(1.12) = 1 - \Phi(-1.12) = 0.8686$

p-value = 1- 0.8686 = 0.1314

<u>13.14% chance</u>

Difficulties with PERT

1. The assumption that the activity durations are independent is difficult to justify.
2. Activity durations may not follow a beta distribution
3. The assumption that the critical path found by CPM will always be the critical path for the project may not be justified.



$A \quad 17/27$

$B \quad 17/27$

$C \quad 12/27$

$D \quad 12/27$